Fluid Dynamics: Physical ideas, the Navier-Stokes equations, and applications to lubrication flows and complex fluids

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A presentation for AP298r
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Outline

- Part I: elementary ideas
  - A role for mechanical ideas
  - Brief picture tour: small to large lengths scales; fast and slow flows; gases and liquids
- Continuum hypothesis: material and transport properties

  Newtonian fluids (and a brief word about *rheology*)

  stress versus rate of strain; pressure and density variations;

  Reynolds number; Navier-Stokes eqns, additional body forces; interfacial tension: statics, interface deformation, gradients
- Part II: Prototypical flows: pressure and shear driven flows; instabilities; oscillatory flows
- Part III: Lubrication and thin film flows
- Part IV: Suspension flows - sedimentation, effective viscosities, an application to biological membranes

From atoms to atmospheres: mechanics in the physical sciences

- classical mechanics
  - particle and rigid body dynamics
- celestial mechanics
  - motion of stars, planets, comets, ...
- quantum mechanics
  - atoms and clusters of atoms
- statistical mechanics
  - properties of large numbers

Isaac Newton
1642–1727

Continuum mechanics:

(materials viewed as continua)

electrodynamics	solid mechanics
thermodynamics	fluid mechanics

A fluid dynamicist's view of the world*

**

Galaxies

***

Snow avalanche

Mathematics

Astrophysics

Engineering

aeronautical
biomedical
chemical
environmental
mechanical

Geophysics

Fluid dynamicist

Chemistry

Biology

Physics

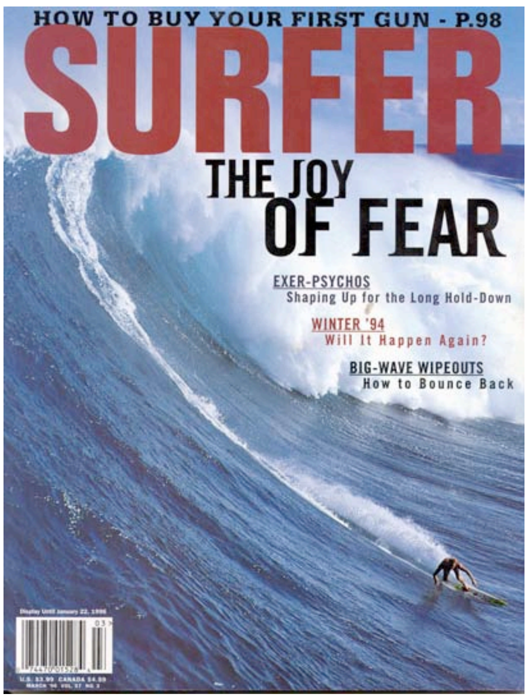
* after theme of H.K. Moffatt
** http://zebu.uoregon.edu/messier.html
*** Courtesy of H. Huppert

Fluid motions occur in many forms around us:

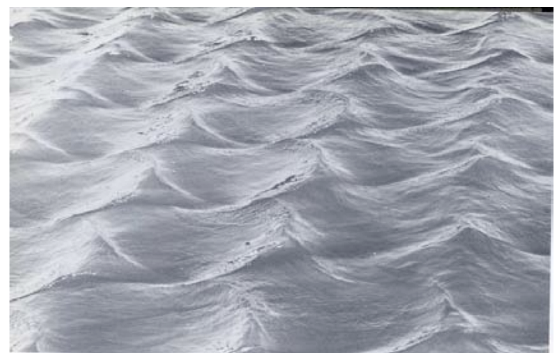
Here is a short tour

(Water) Waves

Big Waves

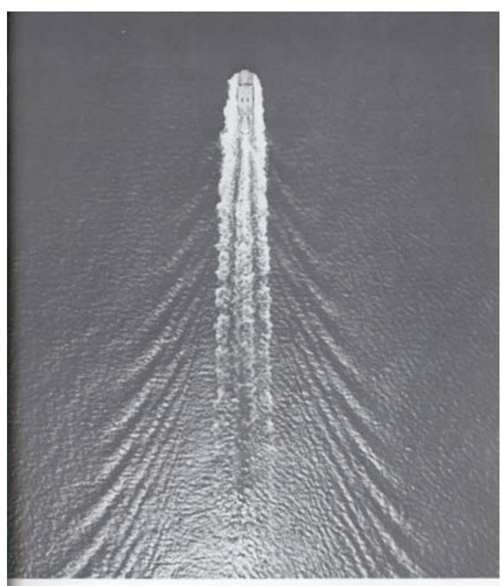


Little waves



194. Spilling breaking waves. This regular three-dimensional pattern, reminiscent of waves in the open sea, has evolved by nonlinear instability from a uniform train of steep two-dimensional Stokes waves. The waves are propagating from left to right with wavelength 0.75 m. *Photograph by Ming-Yang Su*

Ship waves

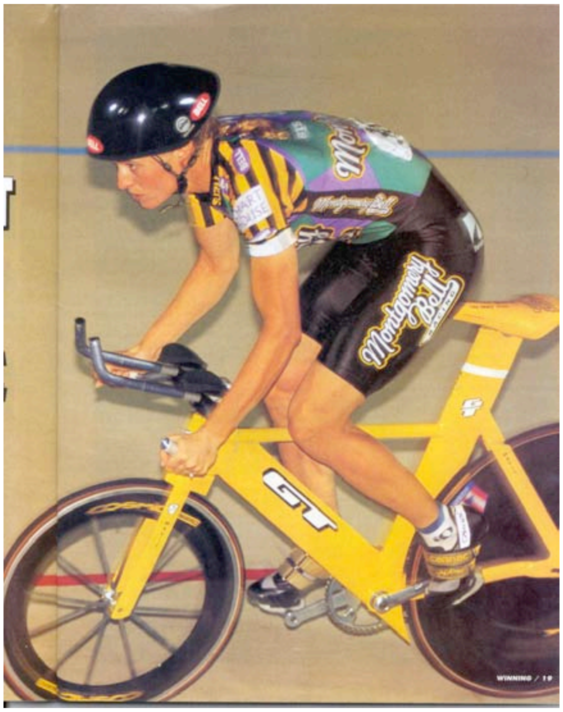


208. Wave pattern of a ship. An aerial photograph from directly overhead shows, away from the breaking wave from the bow and its turbulent wake, the asymptotic pattern deduced by Kelvin in 1887. The waves are confined to an angle of 19½° on either side of the path of the ship, in agreement with the theory, with an effective origin that is displaced approximately one ship length ahead of the bow. *Newman 1970*

Ref.: *An Album of Fluid Motion*, M. Van Dyke

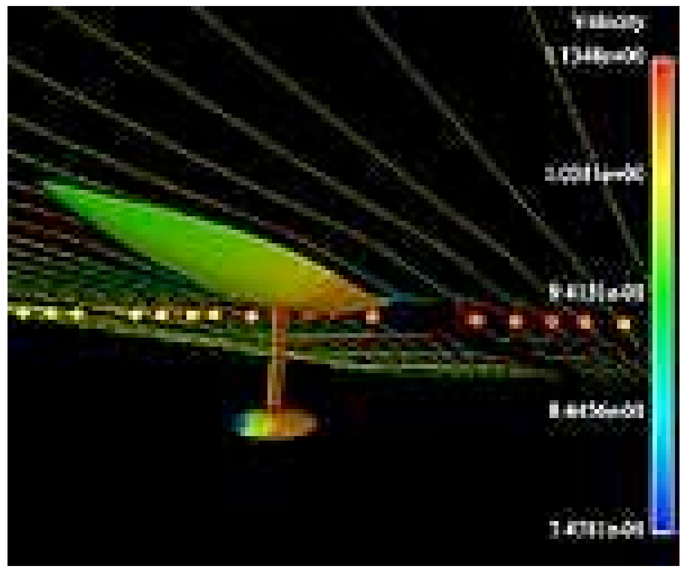
Flow and design in sports

Cycles and cycling



Rebecca Twig, *Winning* Jan. 1996

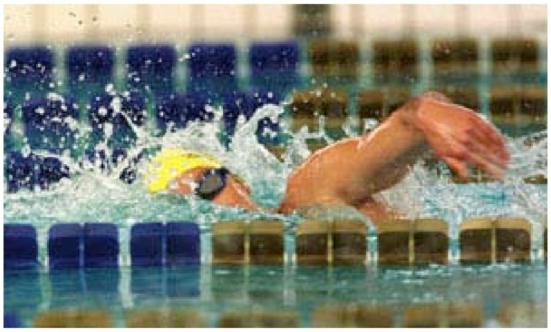
Yacht design and the America's Cup (importance of the keel)



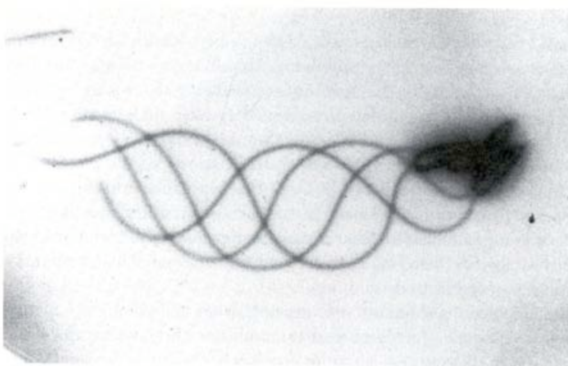
http://www.sgi.com/features/2000/jan/cup/

Bicycling Feb. 1996

Swimming (large and small)



Micro-organisms: flagella, cilia



Four successive positions of the flagellum of a sea urchin sperm (*Lytechinus pictus*), captured by firing four flashes while the camera shutter was open.

Running on water

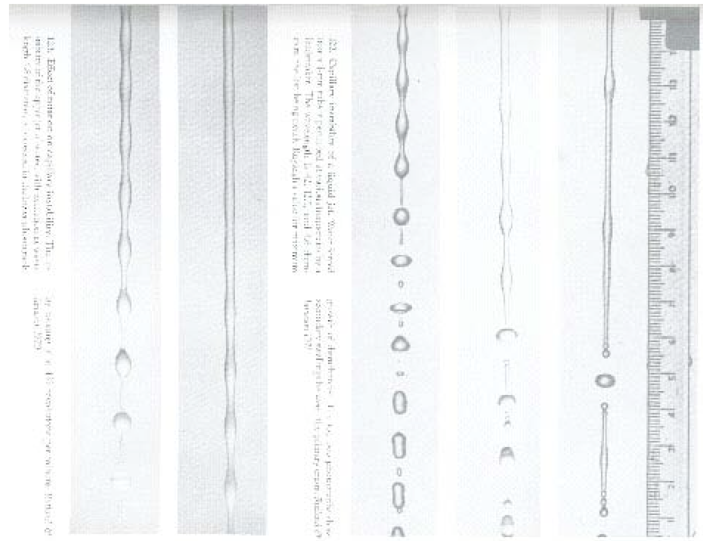
Basilisk or Jesus lizard

Rowing

Speed vs. # of rowers?
T.A. McMahon, *Science* (1971)

Ref: McMahon & Bonner, *On Size and Life*; Alexander *Exploring Biomechanics*

Small fluid drops
(surface tension is important)

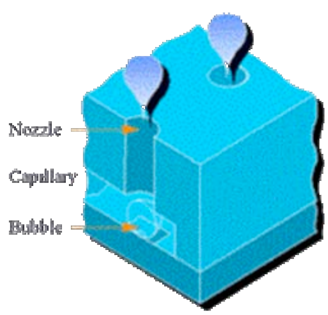


Water issuing from a millimeter-sized nozzle

(3 images on right: different oscillation frequencies given to liquid; ref: Van Dyke, *An Album of Fluid Motion*)

Bubble ink jet printer

(Olivetti)

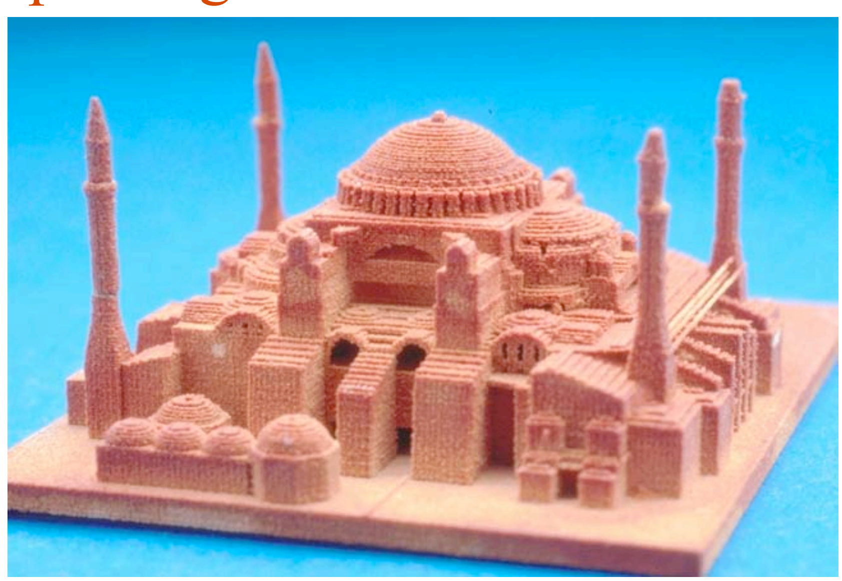


also: deliver reagents to DNA (bio-chip) arrays

Three-dimensional printing -- MIT

(Prof. E. Sachs & colleagues)

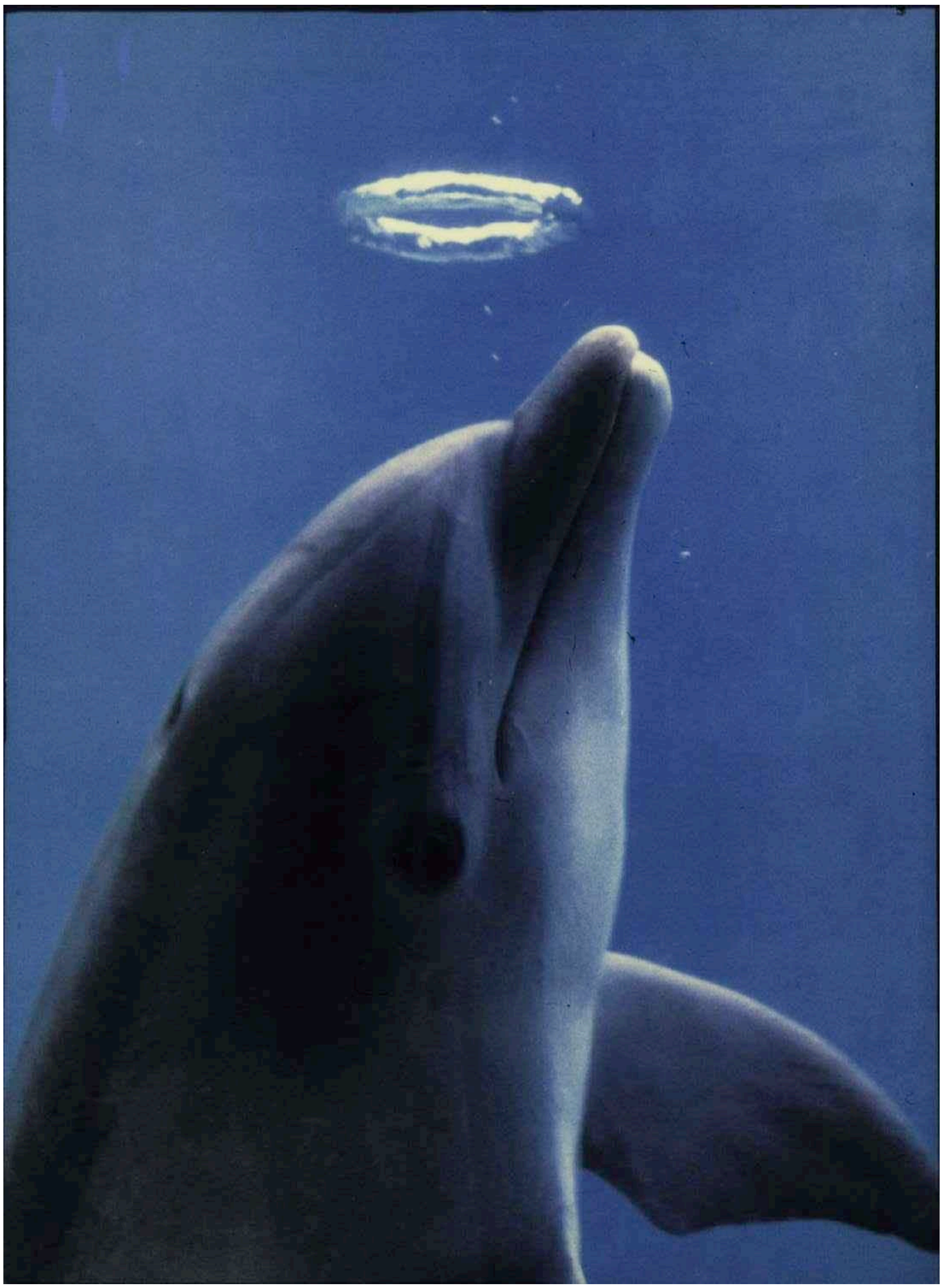
Hagia Sophia ('original' in Istanbul Turkey)



← 5 inches →

…. and a pretty picture ….

A dolphin blowing a toroidal bubble



Elementary Ideas I

- A brief tour of basic elements leading through the governing partial differential equations

- Physical ideas, dimensionless parameters

Elementary Ideas II

2. CONTINUUM HYPOTHESIS:

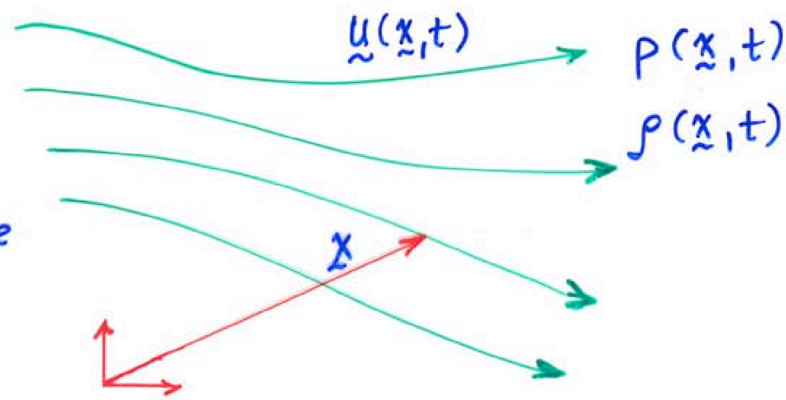
(a) Variables such as pressure, density, temperature, velocity are continuous functions of position.

p = pressure

ρ = density

T = temperature

$\underset{\sim}{u}$ = velocity



(b) cube 1μm on a side: averaging of large numbers

- 3×10^{10} water (ℓ) molecules; 10^{10} benzene molecules
- 10^7 gas molecules at STP
- 10^3 smaller for $\ell \approx 0.1\mu$m ($10^3 \mathring{A}$) on a side.

(c) local thermodynamics equilibrium assumed

$$\Rightarrow \rho = \rho(p, T)$$

Elementary Ideas III

3. MOLECULAR DYNAMICS

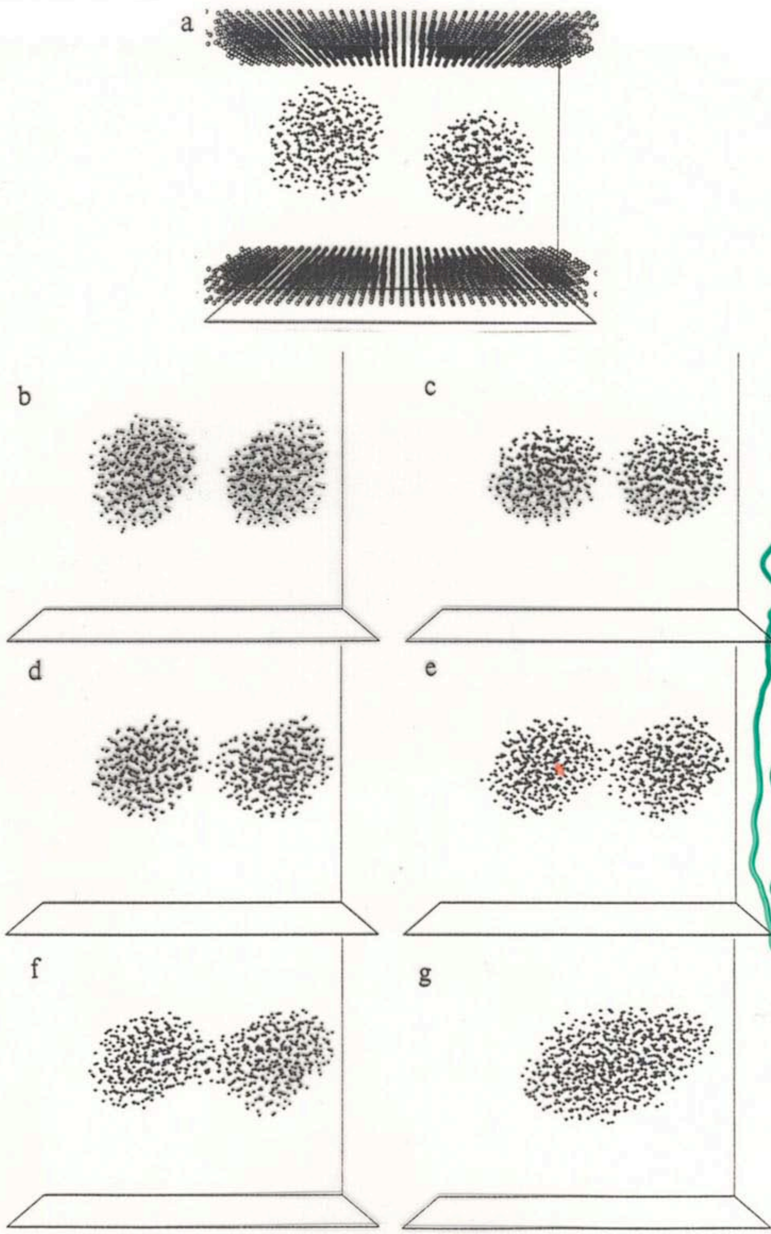


Figure 3 Coalescence of two LJ liquid drops in Couette flow. (*a*) Static atomic configuration of the drops and walls after equilibration at time 20τ. The (immiscible) background fluid is not shown. (*b–g*) Drop configurations under flow at times 75, 96, 98, 100, 102, and 140τ. (Koplik & Banavar 1992)

COALESCENCE OF DROPS DUE TO SHEAR FLOW

TYPICAL CONCLUSION: CALCULATIONS WITH 10^3 molecules (10 per side of a cube!) AGREE WITH CONTINUUM DESCRIPTIONS

Reference: Koplik & Banavar 1995 *Ann. Rev. Fluid Mech.* **27**.

Elementary Ideas IV

THIN FILMS

Experiments on shearing between two molecularly smooth (mica) surfaces separated by thin films of organic liquids.

- Films $>$ 10 molecular diameters can be described in terms of bulk properties.
- Thinner films: molecular ordering, quantization of some properties, "effective" viscosities $> 10^5$ bulk value.
- Film with thickness less than 5 molecular diameters: "solid-like" response.

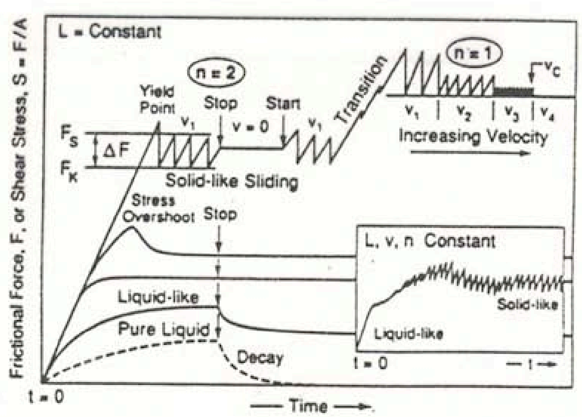


FIG. 5. Frictional forces associated with the different types of sliding modes of Fig. 5. "Pure liquid" sliding occurs with surfaces farther apart than 5–10 σ. "Liquidlike" sliding occurs with configurations as in Figs. 5(d) and 5(e), while "solidlike" sliding is associated with Figs. 5(a), 5(c), and 5(f). With certain liquids the sliding starts by being liquidlike and becomes progressively more solidlike during sliding; this is generally accompanied by a decrease in the film thickness and a "stress overshoot." An example of this given in the inset which shows measured data during an experiment with tetradecane. Note that a single stick–slip occurs over many microns and should not be confused with atomic scale stick–slip (Ref. 38) which may also be occurring but is beyond our resolution.

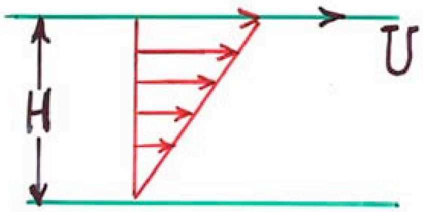
REFERENCE: GEE, McGUIGAN, ISRAELACHVILI & HOMOLA J. CHEM. PHYS. (1990)

Elementary Ideas V

4. Viscosity and Newtonian fluids

VISCOSITY

τ = SHEAR STRESS (FORCE/AREA)



$$\tau = \mu \frac{U}{H}$$

↑ SHEAR VISCOSITY (μ)

Table 1: Viscosity of common liquids

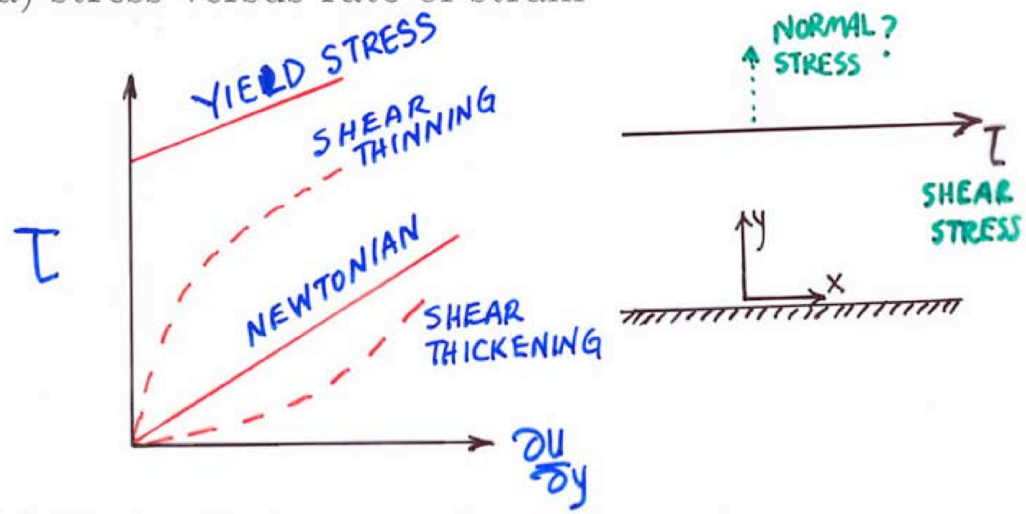
liquid	temperature °C	μ gm/(cm.sec) = Poise	$\nu = \mu/\rho$ (cm^2/sec)
water	10	0.0131	0.0131
water	20	0.01	0.01
water	50	0.0055	0.0056
water	90	0.0031	0.0032
glycerine	20	17.6	14.0
mercury	0	0.014	0.001
lubricating oil	20		4
lubricating oil	40		1
lubricating oil	60		0.3

NOTE: At 20°C, gases have a typical viscosity $\mu \approx 10^{-4}$ gm/(cm.sec), but have a kinematic viscosity $\nu = \mu/\rho \approx 0.1$ cm^2/sec.

Elementary Ideas VI

5. On to the equations of motion

(a) stress versus rate-of-strain



(b) Navier-Stokes equations:

Assume that the material properties ρ and μ are constant (generally an excellent approximation).

MASS BALANCE (CONTINUITY) $\nabla \cdot \mathbf{u} = 0$

NAVIER-STOKES EQUATIONS

LINEAR MOMENTUM BALANCE

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}$$

ACCELERATION — SURFACE FORCES — BODY FORCES

⇝ EACH TERM FORCE/VOLUME

(c) pressure changes accompanying flow

- inertially dominated: $\Delta p = O(\rho U^2)$
- viscously dominated: $\Delta p = O(\mu U/\ell)$

U = typical velocity
ℓ = characteristic length

Elementary Ideas VII

(d) incompressibility $(\nabla \cdot \mathbf{u} = 0)$

variation of density accompanying motion should be small $(\Delta\rho \ll \rho)$

$$\Delta\rho \approx \frac{\partial\rho}{\partial p}\,\Delta p\ , \qquad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

C = SPEED OF SOUND

MACH # = U/c

- inertially dominated flows: $U/c \ll 1$
- viscously dominated flows: $(U/c)^2 \ll \rho U\ell/\mu$

(e) Reynolds number

REYNOLDS NUMBER : $\mathcal{R} = \dfrac{\rho U \ell}{\mu} = \dfrac{U\ell}{\nu}$

Low-Reynolds-number motions: lubrication, film coating, suspensions, MEMS, ... $\Rightarrow$ $\mathbf{0} = -\nabla p + \mu\nabla^2\mathbf{u}$

(f) additional body forces:

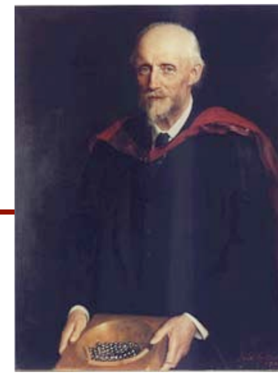
- magnetic: ferrofluids

R.E Rosensweig 1982 Magnetic Fluids. *Scientific American*

R.E. Rosensweig *Ferrohydrodynamics* (Cambridge University Press).

- electric: electric fields and dielectric materials, electrokinetic flows, electrophoresis

Elementary Ideas VIII



Osborne Reynolds
(1842–1912)

The Reynolds number

- Newton's second law:

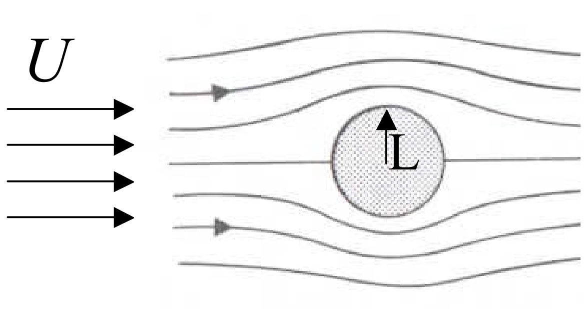
$$mass.acceleration = å\ forces$$

Forces (pressure) acting on fluid to cause motion

Friction from surrounding fluid which resists motion: *viscosity* ( )

High Reynolds number flow

Low Reynolds number flow



ratio of inertial effects to viscous effects in the flow

$$\text{Re} = \frac{rUL}{m}$$

Emphasizes inter-relation of size, speed, viscosity

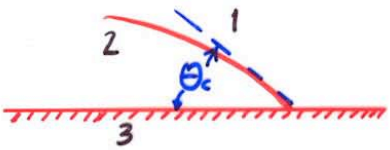
Elementary Ideas IX

5. Interfacial tension γ [force/length or energy/area]

(a) statics

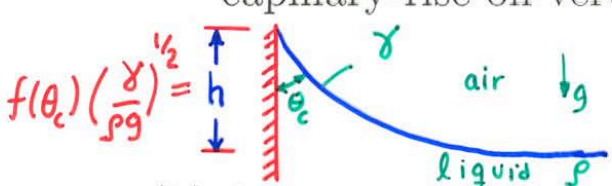
capillary length: $\ell_{cap} = \left(\frac{\gamma}{\rho g}\right)^{\frac{1}{2}}$

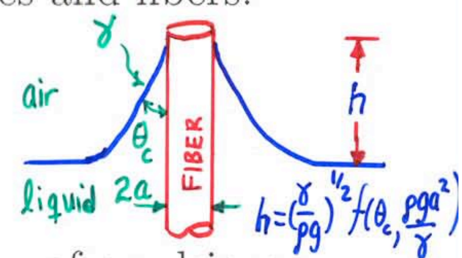
contact angle Θ_c



$\gamma_{12} \cos \theta_c = \gamma_{13} - \gamma_{23}$

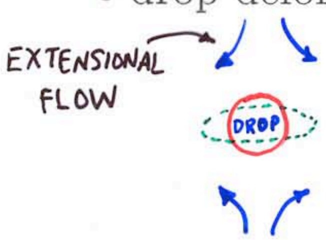
capillary rise on vertical planes and fibers:





(b) dynamics

- drop deformation, formation of emulsions



DEFORMATION $\propto f\left(\frac{\mu G a}{\gamma}\right)$

G= shear rate

a= drop radius

- drop spreading (lecture II)

(c) interfacial tension gradients: Maragoni motions

$\gamma(T)$ $\gamma(c)$

Remark: tangential gradients of c or T give rise to tangential stresses that produce motion.

Quiz 1

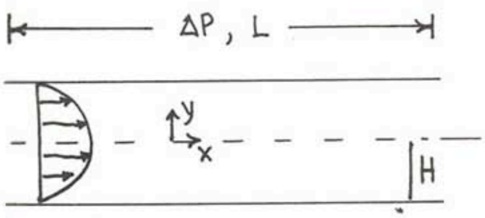
- Consider the rise height of a liquid on a plane.

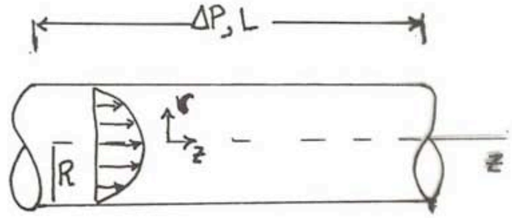
- Use dimensional arguments to show that the rise height is proportional to the capillary length.

PART II: Prototypical Flows I

Steady pressure-driven flow

CHANNEL & PIPE FLOWS





NO-SLIP ON BOUNDARIES

$$u(y) = \frac{H^2}{2}\frac{\Delta p}{\mu L}\left[1 - (y/H)^2\right]$$

$$u(r) = \frac{R^2}{4\mu}\frac{\Delta p}{L}\left[1 - (r/R)^2\right]$$

AVG VELOCITY $\langle U \rangle = \frac{R^2}{8\mu}\frac{\Delta p}{L}$

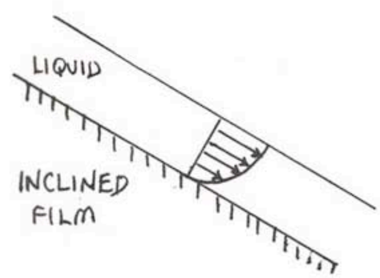
MASS FLOW RATE: $Q = \frac{\pi}{8}\frac{\rho}{\mu} R^4 \frac{\Delta p}{L}$

∴ PARABOLIC (POISEUILLE) VELOCITY PROFILE

APPLICATIONS:

BLOOD FLOW
PIPE FLOW
MEMS

FILM FLOWS:



Prototypical Flows II

Additional effects when the mean free path of the fluid is comparable to the geometric dimensions

GAS FLOW IN A MICROCHANNEL: COMPRESSIBLE FLOW WITH SLIP

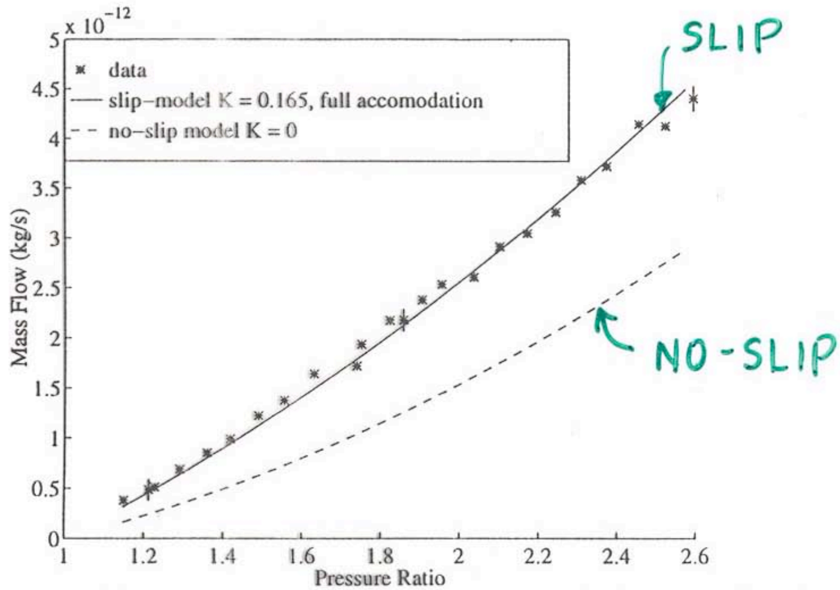


Figure 10: Helium mass flow for 1.33 micron channel, compared with Equation 23. The solid curve is the solution to Equation 23, assuming full tangential momentum accommodation and the dashed curve is the solution to Equation 23 setting $K = 0$ (no-slip solution).

REF: ARKILIC, SCHMIDT & BREUER

Prototypical Flows III

Even simple flows suffer dynamical instabilities!

Osborne Reynolds and the instability of pipe flow.

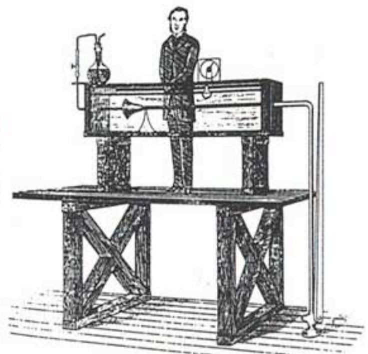


Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper.

$$R = \frac{U \cdot (\text{diameter})}{\nu}$$

LAMINAR

TURBULENT

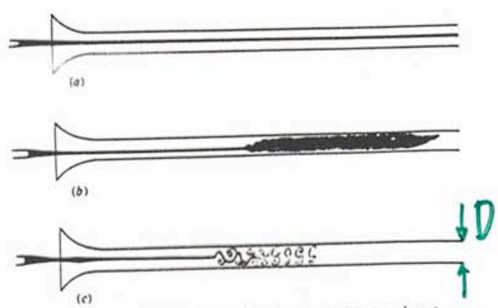


Fig. 9.2. Reynolds's drawings of the flow in his dye experiment.

INCREASING REYNOLDS #

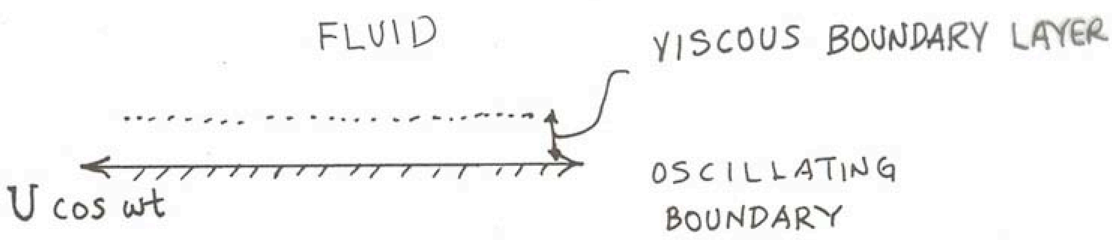
Under "typical conditions", pipe flow becomes unstable when $R = \frac{U \cdot D}{\nu} \gtrsim 2000$.

If (great) care is taken, the flow can be stable at (much) larger Reynolds number (> 50,000).

Ref. D. Acheson

Prototypical Flows IV

OSCILLATORY FLOWS



BOUNDARY LAYER δ OUTSIDE OF WHICH THERE IS ALMOST NO MOTION.

$$\delta \approx \left(\nu/\omega\right)^{1/2}$$

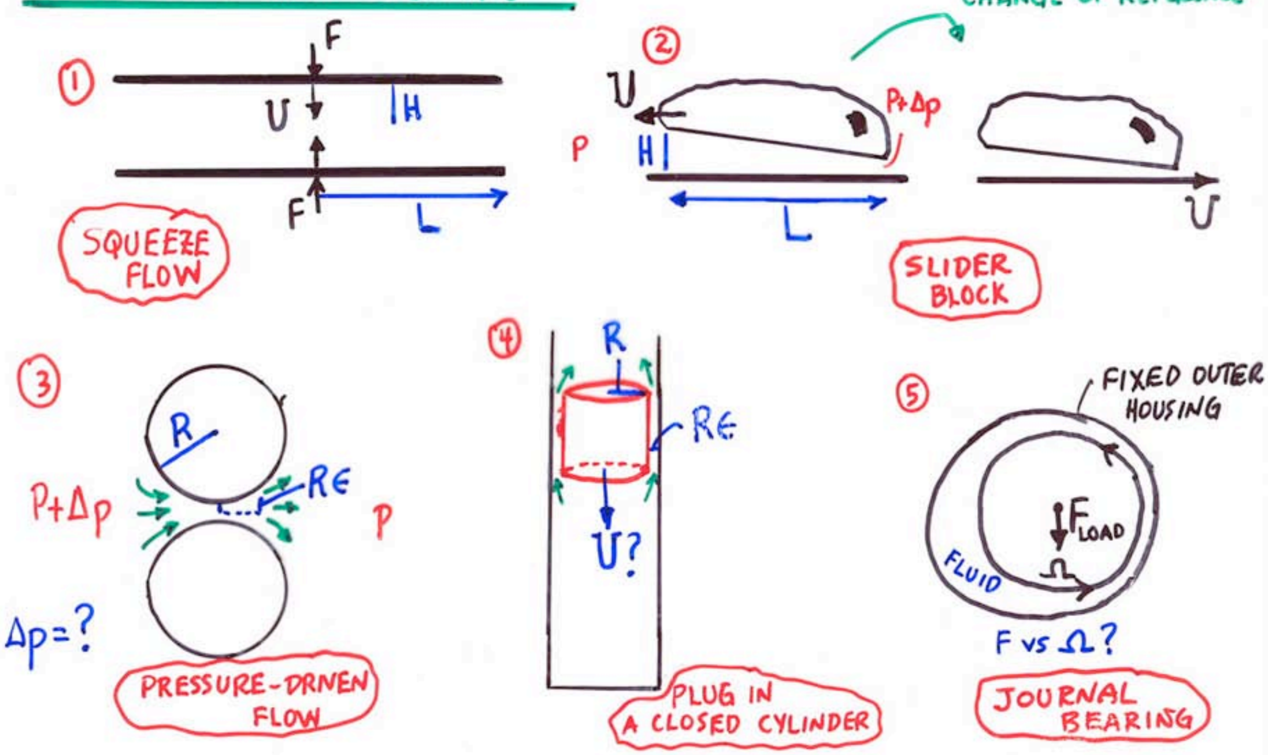
ν = KINEMATIC VISCOSITY

- AS $\nu \uparrow$ THE VISCOUS COUPLING INCREASES.
- AS $\omega \uparrow$ THE BULK KNOWS LESS AND LESS ABOUT THE BOUNDARY MOTION.

Lubrication Flows I

THEME: FLUID MOTIONS CHARACTERIZED BY LONG, NARROW GEOMETRIES.

5 TYPICAL CONFIGURATIONS



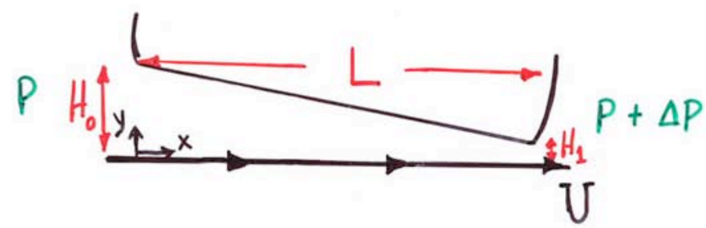
CHARACTERISTIC FEATURES

(i) PRESSURE DROP ALONG THE FLOW DIRECTION

(ii) LARGEST VELOCITY GRADIENTS ACROSS THE FLOW

Lubrication Flows II

LUBRICATION FLOWS : PRESSURE DROP VS VELOCITY



NAVIER-STOKES EQNS: INERTIA IS NEGLIGIBLE

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{u}\right) = -\underset{\sim}{\nabla} p + \mu \nabla^2 \underset{\sim}{u}$$

[NEGLECT BODY FORCES]

NEARLY ONE-DIMENSIONAL FLOW

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\therefore \quad \frac{\Delta p}{L} = O\left(\mu \frac{U}{H_0^2}\right)$$

"long length scale" → L

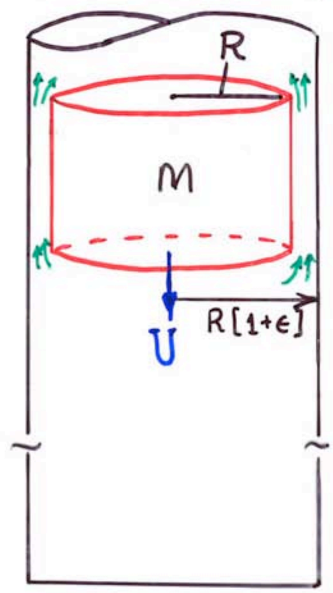
H_0 ← "small length scale"

LARGE PRESSURE VARIATIONS IN NARROW GAPS!

APPLICATIONS OF THIS IDEA EXPLAIN A WIDE RANGE OF LUBRICATION PHENOMENA.

Lubrication Flows III

EXAMPLE: CYLINDRICAL PLUG SEDIMENTING IN A CLOSED TUBE



$\epsilon \ll 1$

ORDER-OF-MAGNITUDE FOR THE FALL SPEED:

U = FALL SPEED

u = TYPICAL FLUID VELOCITY IN THE GAP

MASS CONSERVATION: $U \pi R^2 \approx u \, 2\pi R (R\epsilon)$

LUBRICATION: [PRESSURE DROP] $\Delta P \approx \frac{\mu u L}{(R\epsilon)^2}$

FORCE: $mg = F = \Delta p \cdot \pi R^2 \longrightarrow \boxed{U \propto \epsilon^3}$

[SHEAR STRESSES ON THE SIDES ARE NEGLIGIBLE]

Quiz 2

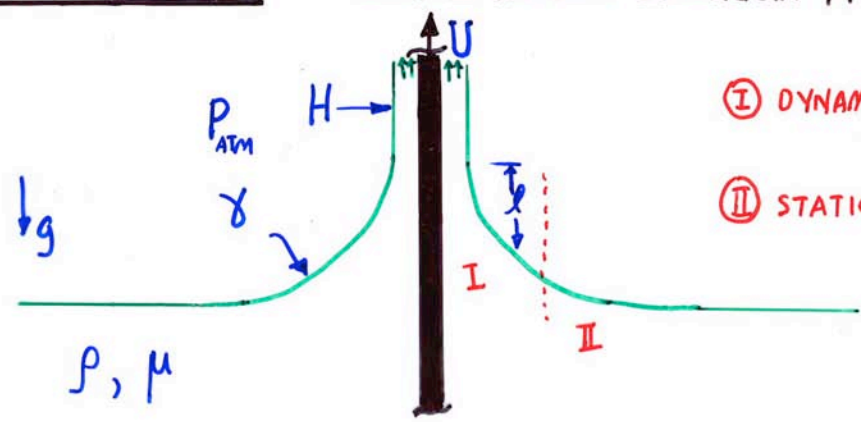
- Consider pressure-driven flow in a rectangular channel of height h and width w with $h << w$.

- Find an approximate expression for the flow rate through the channel.

- If the permeability is the ratio of the $\quad u/(\quad p/L)$, find the permeability of such a rectangular channel.

Lubrication Flows IV

FILM COATING: LANDAU-LEVICH-DERJAGUIN PROBLEM



(I) DYNAMICS: μ, γ

(II) STATICS: γ, g

REGION II: FAR AWAY FROM THE PLATE THE FLUID IS NEARLY STATIC.

DEFORMED STATIC INTERFACE CHARACTERIZED BY: $\ell_{cap} \approx \left(\frac{\gamma}{\rho g}\right)^{1/2}$ CAPILLARY LENGTH

REGION I: SLOWLY VARYING FLOW

'VISCOUS ENTRAINMENT' VS 'CAPILLARY SUCTION'

$$\mu \frac{U}{H^2} \approx \frac{\Delta p}{\ell} \qquad \Delta p \approx \gamma \frac{H}{\ell^2}$$

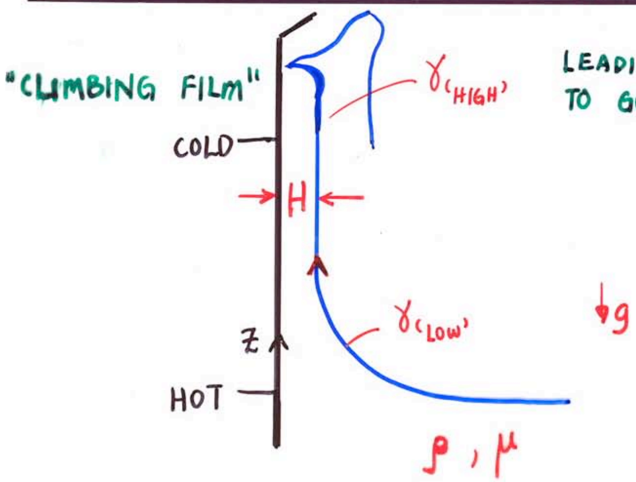
SLOWLY VARYING INTERFACE

"OVERLAP": CURVATURES OF REGIONS I & II AGREE: $\frac{H}{\ell^2} \approx \ell_{cap}^{-1}$ ← DETERMINES ℓ

$$\therefore \quad H \approx \frac{(\mu U)^{2/3}}{(\rho g)^{1/2} \gamma^{1/6}}$$

Lubrication Flows V

FILM COATING DRIVEN BY SURFACE TENSION GRADIENTS



LEADING EDGE OBSERVED TO GO UNSTABLE

FLOW DRIVEN BY SURFACE SHEAR STRESS τ

$$\tau = \left(\frac{d\gamma}{dT}\right) \frac{dT}{dz}$$

$\left(\frac{d\gamma}{dT}\right)$ ↑ MATERIAL PROPERTY

$\frac{dT}{dz}$ ↑ APPLIED TEMPERATURE GRADIENT

LUBRICATION APPROXIMATION: $\frac{\Delta p}{\ell} \approx \mu \frac{u}{h^2} \approx \frac{\tau}{h}$

DYNAMIC REGION CONNECTED TO A STATIC MENISCUS

$$\therefore \; H \approx \frac{\tau^2}{\gamma^{1/2} (\rho g)^{3/2}}$$

FILM THICKNESS $\propto \left(\frac{dT}{dz}\right)^2$

[FANTON, CAZABAT & QUÉRÉ, 1996]

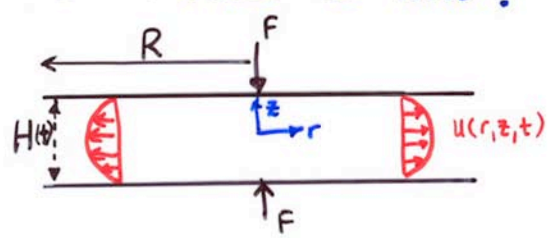
Lubrication Flows VI

Time-dependent geometries

SQUEEZE FLOW BETWEEN TWO DISKS

TWO CIRCULAR DISKS ARE SQUEEZED TOGETHER WITH A CONSTANT FORCE.

SEPARATION DISTANCE VS TIME ?



MASS CONSERVATION $\quad U\pi R^2 \approx u\, 2\pi R H \;\rightarrow\; u = U\frac{R}{H}$

LUBRICATION $\quad \frac{\Delta p}{R} \approx \mu \frac{u}{H^2} \approx \mu \frac{UR}{H^3}$

FORCE $\quad F \simeq (\Delta p)\, \pi R^2 \approx \mu \frac{U R^4}{H^3}$

GAP THICKNESS $\quad U \simeq \frac{dH}{dt} \;\rightarrow\; \frac{dH}{dt} \propto H^3$

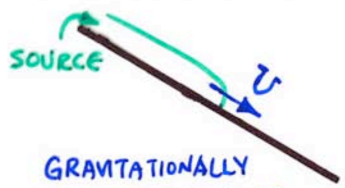
$\therefore \; H \approx t^{-1/2}$ for large times

Lubrication Flows VII

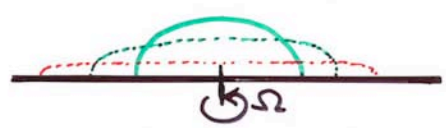
SPREADING FILMS

DYNAMICS OF LIQUIDS SPREADING ON SOLID (OR LIQUID) SUBSTRATES

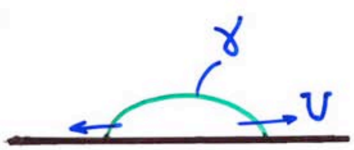
MODEL PROBLEMS



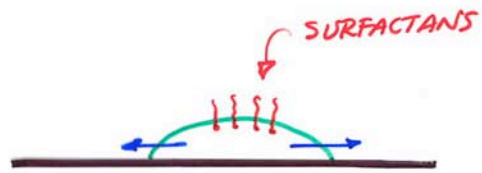
GRAVITATIONALLY DRIVEN SPREADING



COATING OF A SPINNING DISK



SURFACE-TENSION (CAPILLARY) DRIVEN SPREADING



SURFACE TENSION GRADIENTS DRIVE SPREADING

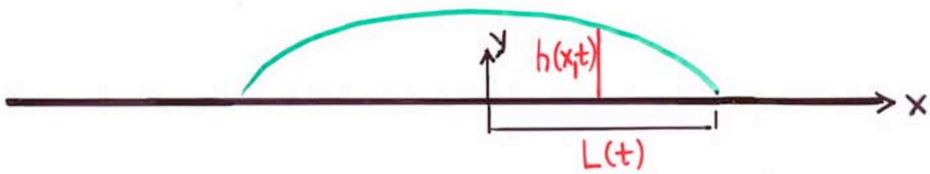
EACH SPREADING CONFIGURATION IS CHARACTERIZED BY A LONG, NARROW REGION OF FLOW.

⇓

LUBRICATION APPROXIMATION

Lubrication Flows VIII

SPREADING RATES



SOME IMPORTANT IDEAS

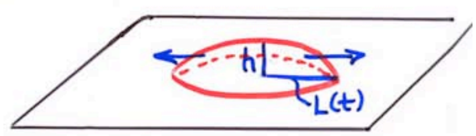
① DETERMINE $L(t)$, $h(x,t)$

TYPICAL VELOCITIES $u \approx \frac{dL}{dt} \approx \frac{L(t)}{t}$

② COMBINE MOMENTUM BALANCE $\left[\frac{\Delta p}{L} \approx \mu \frac{u}{h^2} \right]$

WITH A GLOBAL MASS BALANCE

2D $h \cdot L \approx$ CONSTANT AXISYMMETRIC $hL^2 \approx$ CONSTANT



③ DRIVING FORCES

GRAVITY $\Delta p \approx \rho g h$

CENTRIFUGAL $\Delta p \approx \frac{1}{2} \Omega^2 L^2$

SURFACE TENSION $\Delta p \approx \gamma h / L^2$

[NOT DRIVEN BY SPREADING COEFFICIENT de Gennes 1985]

Lubrication Flows IX

SCALING LAWS

THE EVOLUTION OF THE FILM SHAPE CAN BE PREDICTED BY SOLVING A NONLINEAR PDE.

SPREADING OF A 2D GRAVITY CURRENT

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) \Rightarrow h(x,t)$$

+ global mass conservation

$$L(t) \approx t^{\alpha}$$

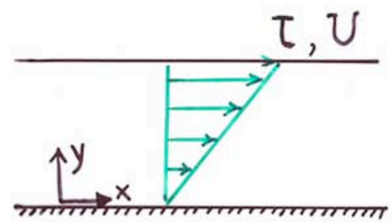
DRIVING FORCE	SPREADING RATE	
GRAVITY	$\alpha = \frac{1}{5}$	2D
	$= \frac{1}{8}$	AXISYMMETRIC
ROTATION (SPIN COATING)	$\alpha = \frac{1}{4}$	AXISYMMETRIC
SURFACE TENSION	$\alpha = \frac{1}{7}$	2D
	$= \frac{1}{10}$ (TANNER'S LAW)	AXISYMMETRIC

Suspension Flows I

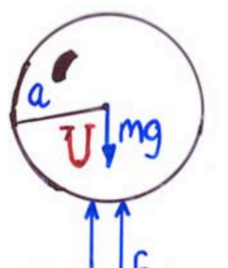
SEDIMENTATION

- DEFINITION OF VISCOSITY

SHEAR STRESS (FORCE/AREA) $= \tau = \mu \dfrac{\partial u}{\partial y}$



- SEDIMENTING SPHERE



$\downarrow \underset{\sim}{g}$

$\underset{\sim}{F}_{external} = \rho_p \dfrac{4\pi a^3}{3} \underset{\sim}{g}$

(ARCHIMEDES) $\quad -\rho \dfrac{4\pi a^3}{3} \underset{\sim}{g} = \underset{\sim}{F}_{BUOYANT} \qquad \underset{\sim}{F}_{HYDRO}$

order of magnitude $\quad \underset{\sim}{F}_{HYDRO} \simeq \text{SHEAR STRESS} \cdot \text{AREA} \approx -\dfrac{\mu U}{a} 4\pi a^2 \approx -4\pi \mu a \underset{\sim}{U}$

STOKES: $\quad \underset{\sim}{F}_{HYDRO} = -6\pi \mu a \underset{\sim}{U}$

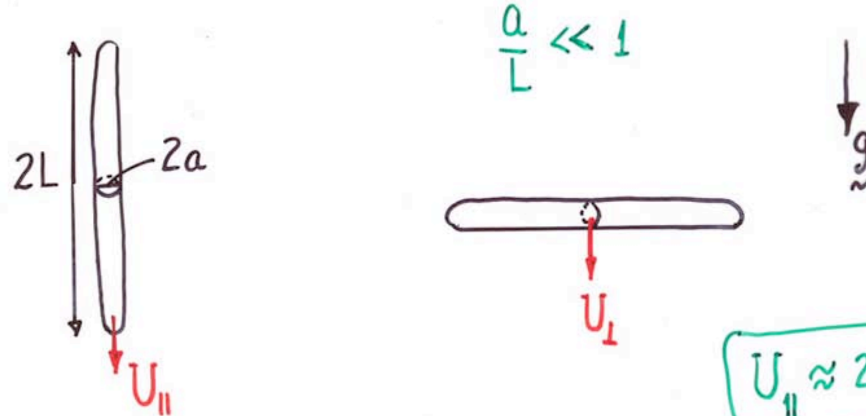
$$\therefore \quad \boxed{\underset{\sim}{U} = \frac{2a^2(\rho_p - \rho)\underset{\sim}{g}}{9\mu}}$$

← a^2 dependence

Suspension Flows II

SEDIMENTATION: SLENDER PARTICLES

$$\frac{a}{L} \ll 1$$



$$U_{\parallel} \approx 2 U_{\perp}$$

$$\text{Force} = \frac{\text{force}}{\text{length}} \cdot \text{length}$$

$$\approx O(\mu U \cdot 2L) \qquad \text{(DIMENSIONAL ARGUMENT)}$$

INDEPENDENT OF a ?! ⇒ THE PROBLEM WITH A PURELY DIMENSIONAL ARGUMENT!

LOW REYNOLDS NUMBERS: $Re = \frac{Ua}{\nu}$

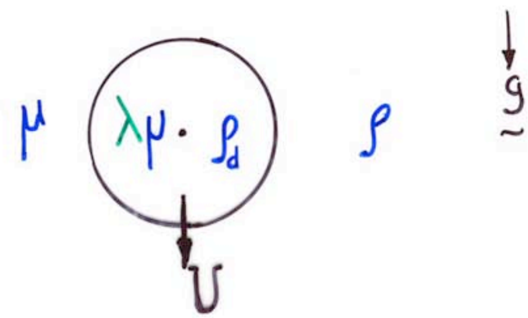
(i) $1 \ll \frac{L}{a} \ll Re^{-1}$ $\qquad F \approx \frac{2\pi \mu U L}{\ln(L/a)}$

(ii) $1 \ll Re^{-1} \ll \frac{L}{a}$ $\qquad F \approx \frac{2\pi \mu U L}{\ln(Re^{-1})}$

APPLICATION TO THE PROPULSION OF SWIMMING MICROORGANISMS. [BERG & PURCELL]

Suspension Flows III

SEDIMENTATION VELOCITY OF SMALL DROPS



$$\underset{\sim}{F}_{HYDRO} = -4\pi a \mu \underset{\sim}{U} \left(\frac{1 + \frac{3}{2}\lambda}{1 + \lambda} \right)$$

RYBCZYŃSKI (1911)
HADAMARD (1912)

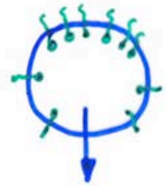
$$\underset{\sim}{U} = \frac{(\rho_d - \rho) a^2 \underset{\sim}{g}}{3\mu} \frac{(1 + \lambda)}{(1 + \frac{3}{2}\lambda)}$$

$$\frac{U_{\lambda=0}}{U_{\lambda=\infty}} = \frac{2}{3} \text{ only}$$

CLEAN INTERFACES

NOTE: SURFACTANTS CAN HAVE A SIGNIFICANT INFLUENCE. FREQUENTLY, SMALL DROPS RISE LIKE RIGID SPHERES.

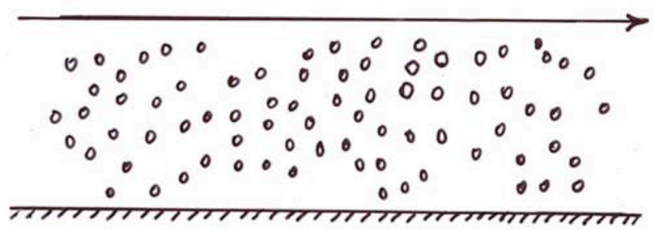
SEDIMENTS ALMOST LIKE A RIGID SPHERE



SURFACTANTS PRODUCE SURFACE TENSION GRADIENTS WHICH PRODUCE A NEARLY RIGID INTERFACE.

Suspension Flows IV

EFFECTIVE VISCOSITY OF A SUSPENSION



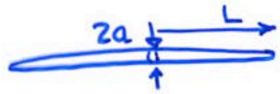
- DILUTE SUSPENSION $\phi \ll 1$, ϕ = VOLUME FRACTION OF PARTICLES
- ON AVERAGE EXPECT A RESISTANCE AS IF THE MEDIUM WERE HOMOGENEOUS WITH AN EFFECTIVE VISCOSITY

$$\mu_{eff} = \mu \left[1 + \frac{5}{2} \phi \right]$$ (EINSTEIN 1906)

- IF THE SUSPENDED PARTICLES ARE DROPLETS (SPHERICAL) OF VISCOSITY $\lambda \mu$, THEN

$$\mu_{eff} = \mu \left[1 + \phi \left(\frac{1 + \frac{5}{2}\lambda}{1 + \lambda} \right) \right]$$ (TAYLOR 1932)

- SLENDER RODS :



"EFFECTIVE VISCOSITY" $\propto \phi (L/a)^2$

SIGNIFICANT EVEN FOR $\phi \ll 1$

(BATCHELOR 1970)

Suspension Flows IV

Brownian motion and diffusion:
The Stokes-Einstein equation

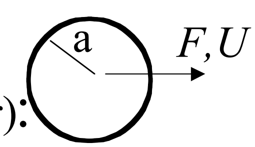
- A typical diffusive displacement in time are linked by (distance)^2 D_t .

- Translation diffusion of spherical particles
- Einstein: related thermal fluctuations to mean square displacement; with resistivity: = force/velocity

where = F/U

- Stokes: = 6 a =fluid viscosity

Stokes-Einstein equation

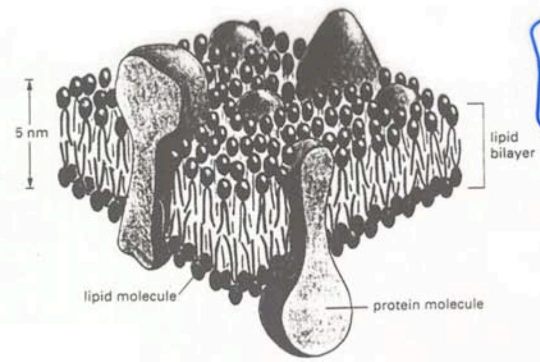


- Typical magnitudes (small molecules in water):

… can also investigate other shapes, rotational diffusion

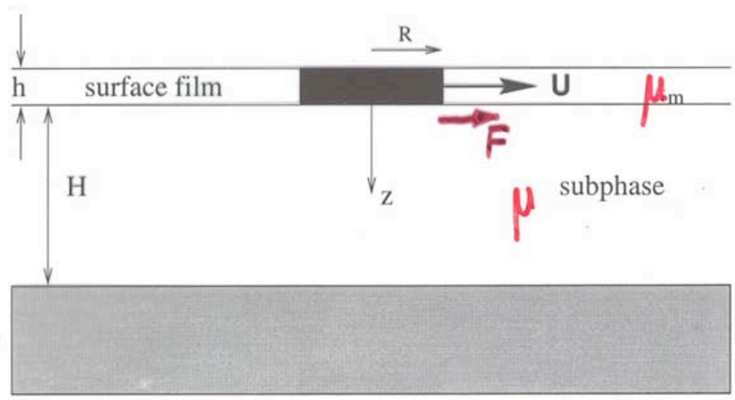
Suspension Flows VI

CELL MEMBRANE
⟿
THE PHYSICAL SYSTEM



DIFFUSION OF PARTICLES IN MEMBRANES

SUPPORTED MEMBRANES
[SACKMANN 1996, SCIENCE]



THE HYDRODYNAMIC MODEL
(SAFFMAN-DELBRÜCK, 1975,6)

TRANSLATIONAL DIFFUSION COEFFICIENT

$$D_T = \frac{k_B T}{(F/U)}$$

Suspension Flows VII

PARTICLE MOTION IN MEMBRANES

PREDICTED TRANSLATIONAL DIFFUSION COEFFICIENTS FOLLOW FROM

$$D = \frac{k_B T}{(F/U)}$$

HYDRODYNAMIC MODEL ACCOUNTING FOR SURROUNDING FLUID AND A NEARBY RIGID BOUNDARY SUGGEST

(SAFFMAN) 1976

$$F = -\frac{4\pi\mu R U}{\Lambda\left[\ln(2/\Lambda) - \gamma\right]} \qquad (H \to \infty) \qquad \gamma \simeq 0.57$$

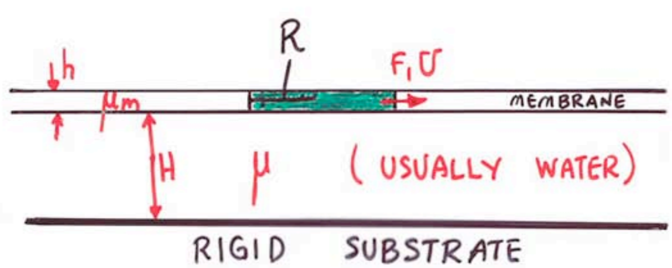
(STONE & AJDARI) 1998

$$F = -\frac{4\pi\mu R U}{\Lambda\left[\ln\left(\frac{\Lambda R}{4H}\right)^{-1/2} - \gamma\right]} \qquad (\text{FINITE } R/H)$$

$$\Lambda = \frac{\mu R}{\mu_m h}$$

MATERIAL PARAMETER CHARACTERIZING VISCOUS RESISTANCE

[SEE ALSO EVANS & SACKMANN (1988)]



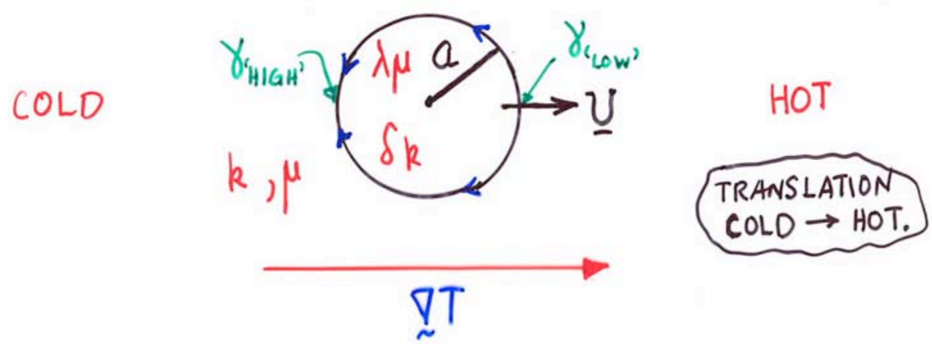
Ref. Stone & Ajdari 1996

Marangoni Flows:
Surface-driven motions

THERMOPHORESIS

SMALL BUBBLES (OR DROPS) IN A LIQUID ARE OBSERVED TO TRANSLATE IN A TEMPERATURE GRADIENT.

SURFACE TENSION $\gamma(T)$



$$\underset{\sim}{U} = -\frac{2a\,(d\gamma/dT)}{(2+\delta)\,(2+3\lambda)\,\mu}\,\underset{\sim}{\nabla} T$$

(δ: conductivity ratio; λ: viscosity ratio)

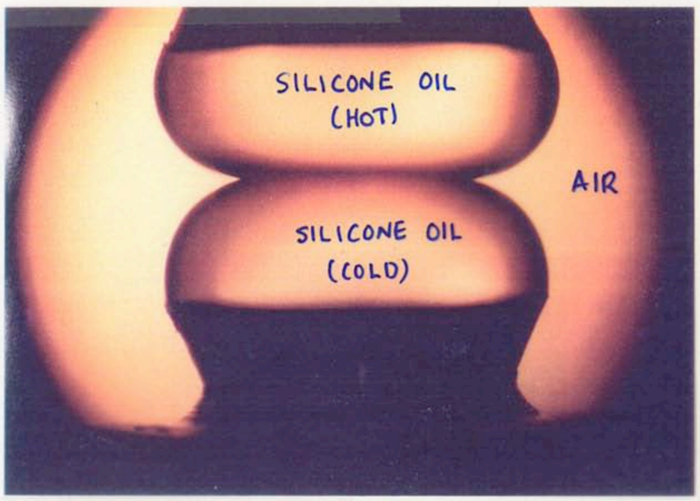
Typically, $\frac{d\gamma}{dT} < 0$

[YOUNG, BLOCK & GOLDSTEIN 1959]

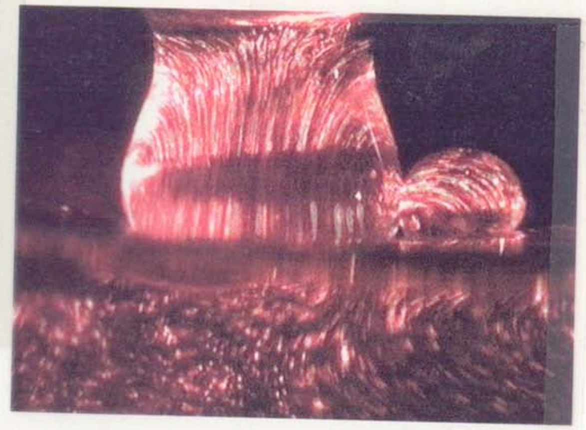
→ MAY DOMINATE BUOYANCY-DRIVEN MOTIONS FOR $a < 10\ \mu m$.

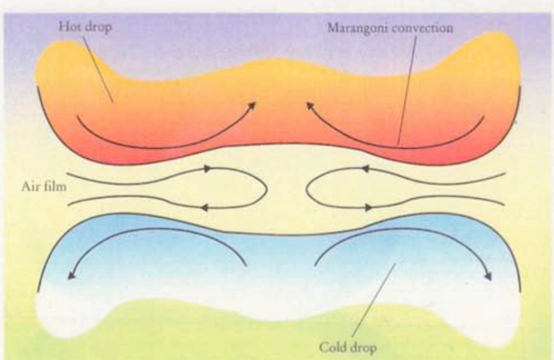
NOTE: AN 'EXPLANATION' BASED UPON BROWNIAN MOTION AND THERMALLY ENHANCED COLLISIONS WOULD PREDICT THAT THE DROP TRANSLATES IN THE DIRECTION HOT → COLD.

More on thermally-driven flows



TWO DROPS AT DIFFERENT TEMPERATURES MAINTAINED IN NEAR CONTACT WITHOUT COALESCING



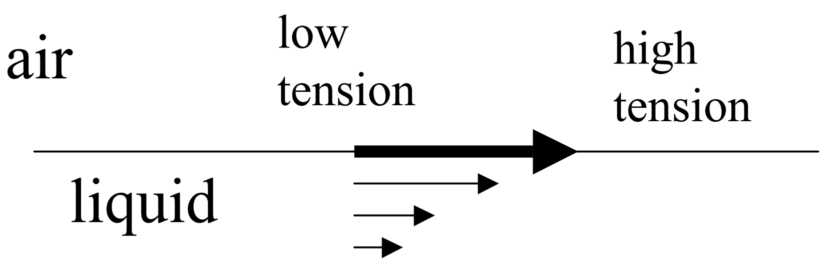


THERMALLY DRIVEN LUBRICATION FLOW WHICH MAINTAINS SEPARATION OF THE INTERFACES

REFERENCE: DELL'AVERSANA & NEITZEL
PHYSICS TODAY, JANUARY 1998

Gradients in surface tension: Marangoni stresses

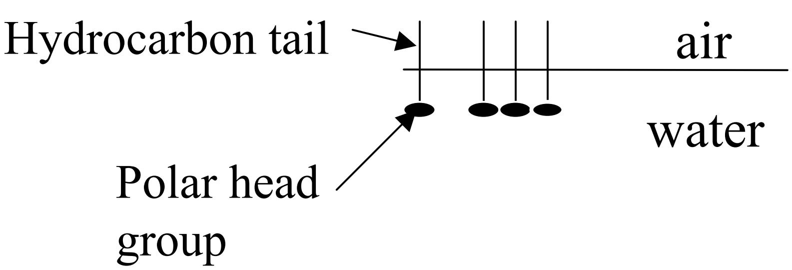
- Local value of surface tension is altered by change of temperature or surfactant concentration



Fluid dragged from low to high tension

- Contaminants typically lower surface tension
- Example: alcohol and water

surfactants: amphiphilic molecules



Courtesy of Professor Maria Teresa Aristodemo, Florence, and Dr. Raffaele Savino, Naples

Gradients in surface tension: Marangoni stresses

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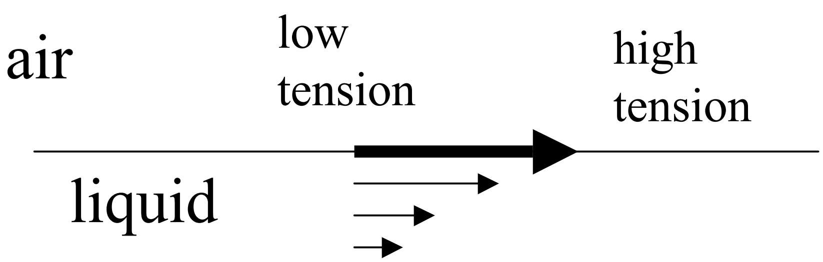
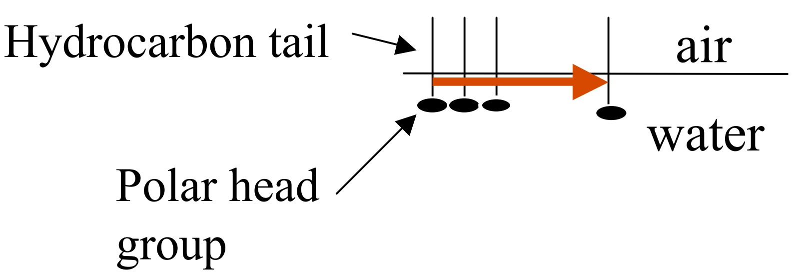


Fluid dragged from low to high tension

- Contaminants typically lower surface tension
- Example: alcohol and water

surfactants: amphiphilic molecules (soap)



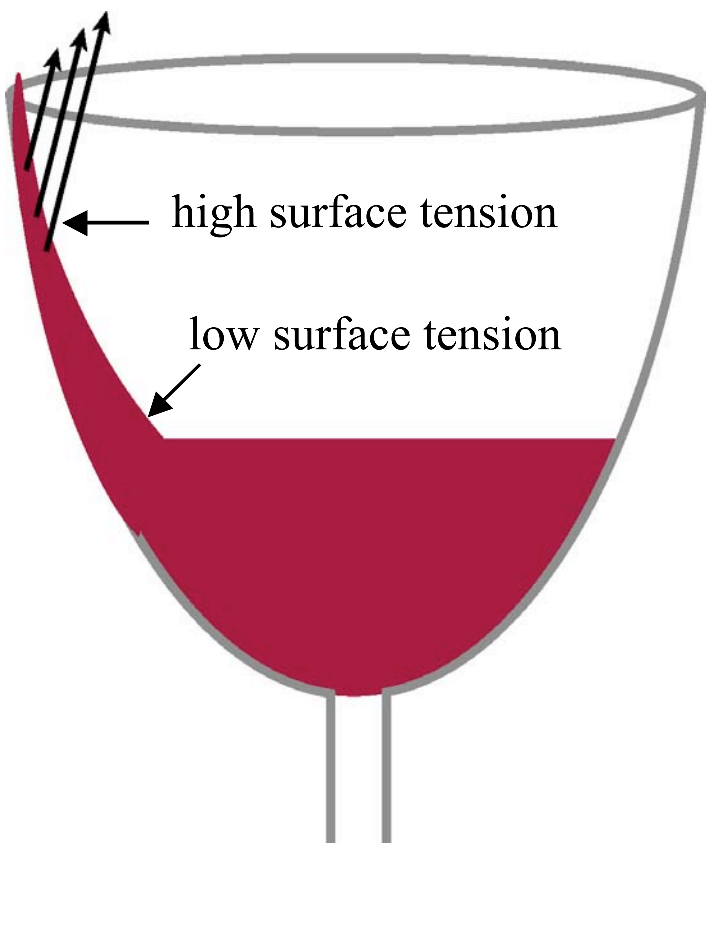
Courtesy of Professor Maria Teresa Aristodemo, Florence, and Dr. Raffaele Savino, Naples

Carlo Marangoni (1840–1925)

Wine tears

An example of the Marangoni effect

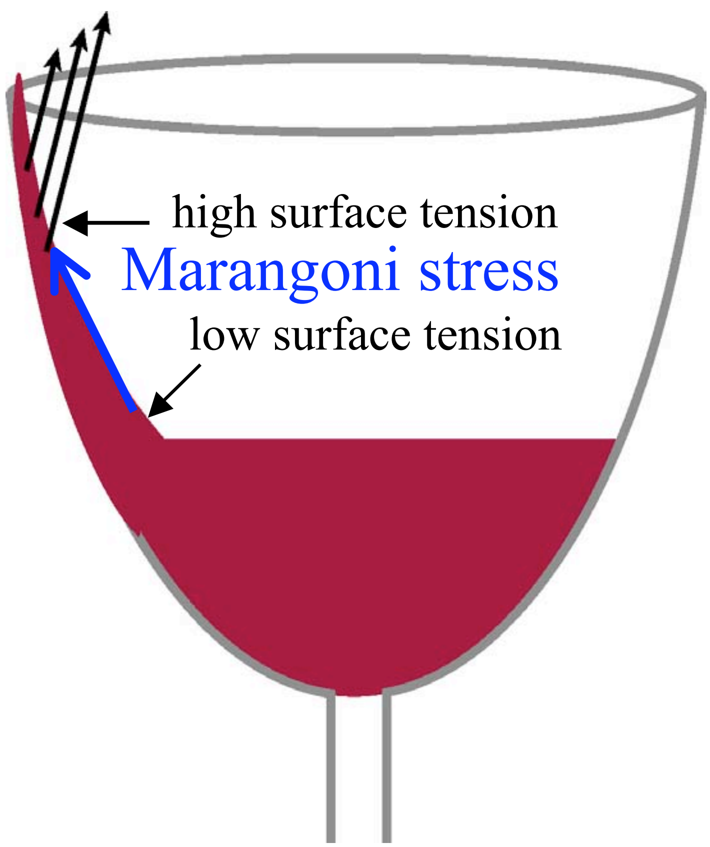
Evaporation from thin film



Wine tears

An example of the Marangoni effect

Evaporation from thin film



In fact: an improved history

- Fluid motions due to gradients in surface were first properly described by James Thomson in 1855

  On certain curious motions observable at the surfaces of wine and other alcoholic liquors

- James Thomson was the older brother of William Thomson (who will appear later in the talk)

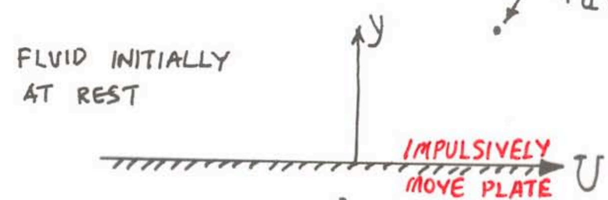
Conclusions

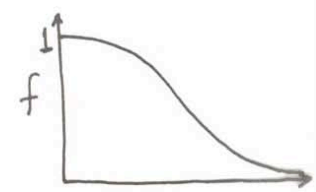
- Continuum descriptions of fluid-like systems begin with momentum statement involving stress (Cauchy equation)
- For Newtonian fluids the starting point is the Navier-Stokes equations which is commonly studied assuming the density and viscosity are constant
- Common geometric configurations, including thin films, are well studied and ammenable to analysis
- Many common features among areas of complex fluids, suspensions, lubricating films, etc.

TIME-DEPENDENT EVOLUTION OF FLOWS

① FLUID RESPONDS ON A "DIFFUSIVE" TIME SCALE WHEN A NEARBY BOUNDARY IS MOVED.

$$u(y,t) = U f\left(\frac{y}{\sqrt{\nu t}}\right)$$



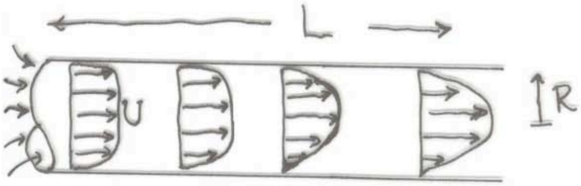


$\frac{y}{\sqrt{\nu t}}$

∴ DIFFUSION TIME $\propto \ell^2/\nu$

DIFFUSION DISTANCE $\propto (\nu\tau)^{1/2}$

② ENTRANCE LENGTH : DISTANCE ALONG A PIPE TO ESTABLISH A PARABOLIC VELOCITY PROFILE



REYNOLDS #

$$R \sim (\nu \cdot \text{time})^{1/2} \approx (\nu L/U)^{1/2} \rightarrow \frac{L}{R} \propto \frac{RU}{\nu}$$

EXPERIMENT $\rightarrow$ $L \approx 0.1(Re)R$

HIGH REYNOLDS NUMBER FLOWS

① CHANGE IN PRESSURE ACCOMPANYING HIGH SPEED FLOW $\left[\mathcal{R} = \frac{U\ell}{\nu} \gg 1 \right]$

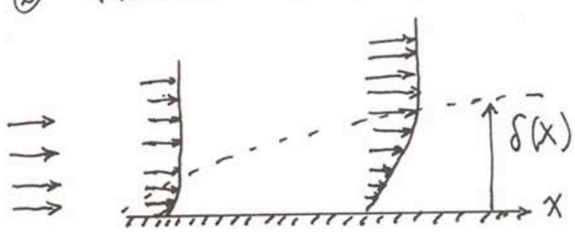
$$\Delta p \propto \rho U^2$$

FORCE ON OBJECT $\propto \rho U^2 \ell^2$

(flow separation)

⇒ LIFT FOR FLYING : SEE THE "GREAT FLIGHT DIAGRAM"

② VISCOUS BOUNDARY LAYER:



$$\rho \frac{U^2}{x} \approx \mu \frac{U}{\delta^2}$$

boundary layer thickness $\equiv \delta(x) \propto \left(\frac{\nu x}{U} \right)^{1/2}$

$\left[\text{AGAIN } \nu^{1/2} \right]$

"THE GREAT FLIGHT DIAGRAM"

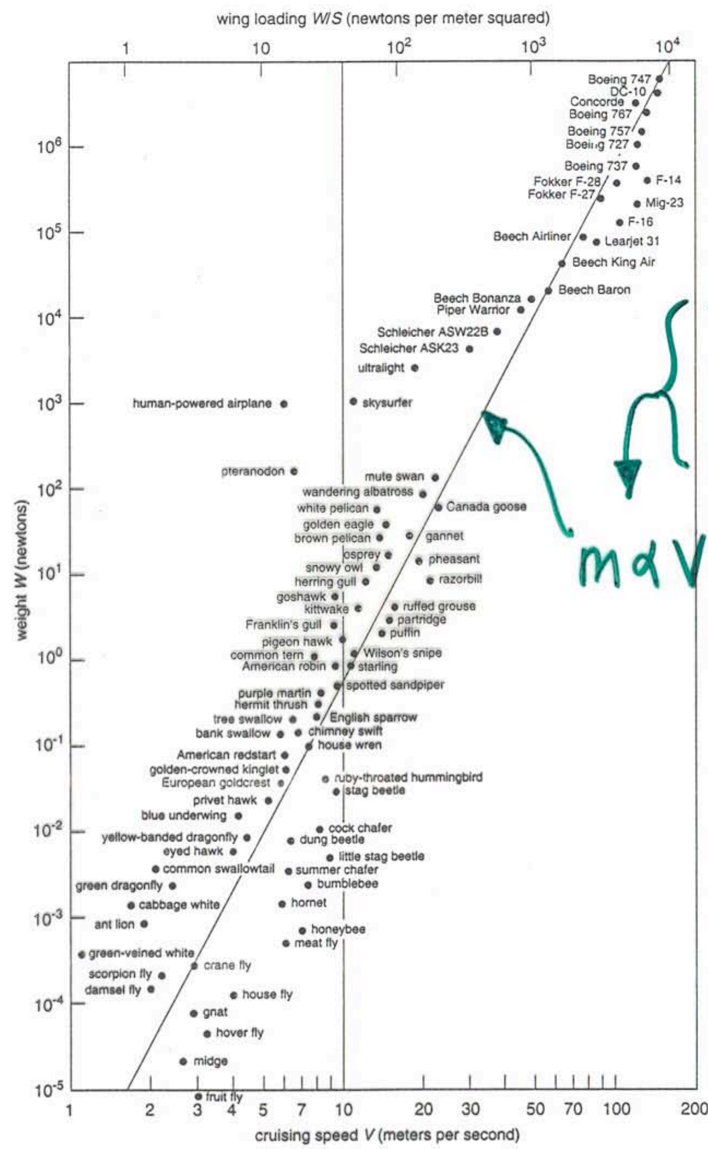


Figure 2 The Great Flight Diagram. The scale for cruising speed (horizontal axis) is based on equation 2. The vertical line represents 10 meters per second (22 miles per hour).

m = mass
V = speed
L = body dimension

$V^2 L^2 \propto m$

$L \propto m^{1/3}$

$m \propto V^6$

REFERENCE: H. TENNEKES "THE SIMPLE SCIENCE OF FLIGHT"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ADVANCEMENT OF
SCIENCE
SCIENCE
8 JULY 1994
VOL. 265 • PAGES 157–288
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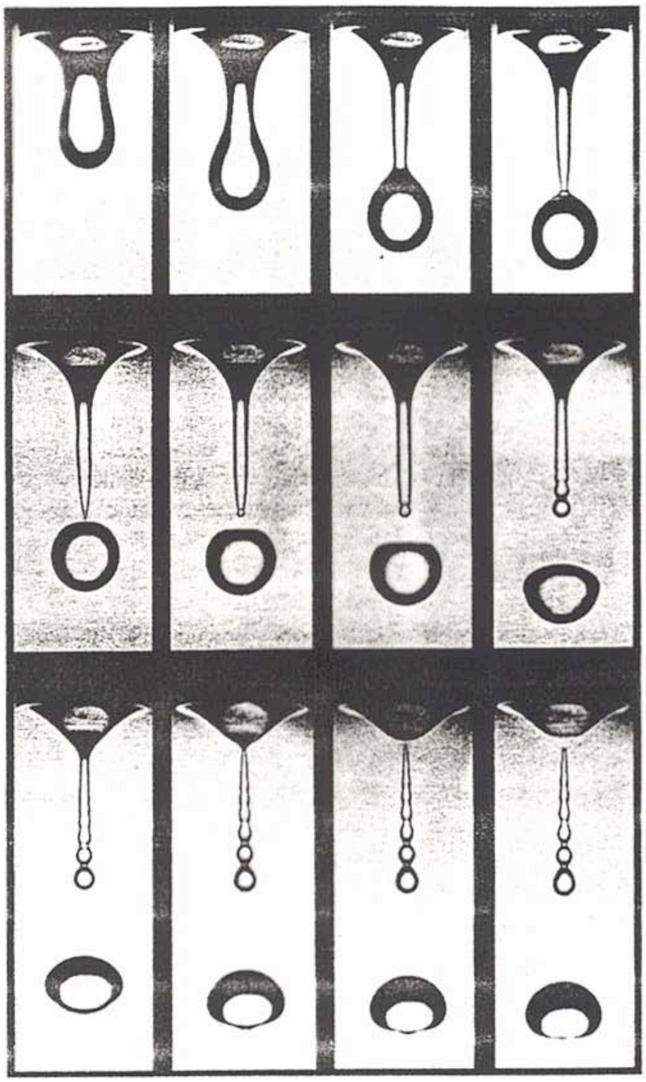

FIG. 6. A sequence of pictures of a water drop falling from a circular plate 1.25 cm in diameter (Shi, Brenner, and Nagel, 1994). The total time elapsed during the whole sequence is about 0.1 s. Reprinted with permission. © American Association for the Advancement of Science.

FIG. 7. A drop of a glycerol and water mixture, 100 times as viscous as water, falling from a nozzle 1.5 mm in diameter. As opposed to the case of water, a long neck is produced (Shi, Brenner, and Nagel, 1994). Reprinted with permission. © American Association for the Advancement of Science.