A viscoelastic beam theory of polymer jets with application to rotary jet spinning

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A B S T R A C T

Complex deformation of a polymer jet appears in many manufacturing processes, such as 3D printing, electrospinning, blow spinning, and rotary jet spinning. In these applications, a polymer melt or solution is first extruded through an orifice and forms a jet. The polymer jet is then dynamically deformed until the polymer solidifies. The final product is strongly affected by the deformation of the polymer jet. And the deformation is strongly affected by the viscoelasticity of the polymer. Here we develop a beam theory to incorporate both the nonlinear viscoelasticity and the bending/twisting stiffness of a polymer jet. As a demonstration, we study the formation of a polymer fiber under strong centrifugal force, a fiber manufacturing process known as rotary jet spinning.

1. Introduction

Extruding polymer melt and solution through an orifice forms a polymer jet. Polymer jets are involved in many manufacturing processes, such as plastic extrusion [1], conventional dry/wet/melt spinning [2,3], 3D printing [4], electrospinning [5,6], blow spinning [7,8], and rotary jet spinning [9–14]. In many applications, the polymer jet undergoes complex deformation before it solidifies. For example, in 3D printing, the coiling of the polymer jet is harnessed to print complex patterns [15]. In electrospinning, the charged jet undergoes chaotic whipping motion which elongates the jet into a nanofiber [16]. In blow spinning, the polymer jet flaps in the high-speed airflow during elongation [17,18]. In rotary jet spinning, the polymer jet is elongated by centrifugal force where bending is induced by Coriolis force and air-drag [9,19]. In these applications, the deformation history of the polymer jet determines the geometry and the microstructure of the final product. Understanding how the processing parameters affect the deformation of the polymer jet is crucial to the development of the manufacturing processes. The search for such understanding motivates the modeling of polymer jets.

In the existing literature, a polymer jet is often modeled as a thin string with no bending stiffness or twisting stiffness. The string model is popular for its mathematical simplicity [17,19–27]. However, neglecting bending and twisting fail the modeling in certain applications. For example, in 3D printing, a polymer jet must be under compression to coil. A string model is impossible to predict coiling under compression [28,29]. Such coiling is also observed in electrospinning near the collector [30]. As another example, during rotatory jet spinning, the polymer jet may be acutely bended near the orifice. The string model is known to diverge in this case [31–34]. For these applications, a polymer jet must be modeled as a beam with finite stiffness towards bending and twisting to correctly predict the deformation. However, since the beam models are mathematically more demanding than the string models, existing beam models are limited to simple material behaviors, such as linear viscosity [32,35,36], linear elasticity [37,38], nonlinear elasticity [39–41], or linear viscoelasticity [42–44]. A beam theory of more complex material behavior such as nonlinear viscoelasticity remains lacking. On the other hand, nonlinear viscoelasticity is often prominent for polymer jets, which are polymer solution or polymer melts undergoing large deformation. As a result, manufacturing processes like 3D printing and rotary jet spinning cannot be accurately modeled in lack of a beam model of nonlinearly viscoelastic material.

In this paper, we develop a beam theory that incorporates nonlinear viscoelasticity. Following the classical Euler–Bernoulli beam theory [45], we assume that a flat cross-section normal to the centerline to remain flat and normal, and approximate the deformation field to the first order of the thickness of the jet. In addition, we enforce the material incompressibility to the first order. The kinematics of finite deformation is derived following these assumptions. The kinematics is combined with a common nonlinear viscoelastic material model, the Oldroyd-B model [46,
3. Deformation of a polymer jet

3.1. Reference state

The deformation of a material point is defined relative to a reference state. We identify the reference state of each material point as the state when the material point passes the orifice. The reference state reflects the loading history of a material point before the jet exits the orifice. Two material points in the jet may have different reference states, depending on the time and location that the material point passed through the orifice and the boundary condition at the orifice. Once the reference state is determined, we define the deformation gradient \( \mathbf{F} \) as the linear map of material vectors from the reference state to the current state following the common practice in continuum mechanics [48].

3.2. Kinematic assumptions

A polymer jet is a 3D body. The accurate kinematics of the jet involves a 3D deformation field. A beam theory approximates the 3D deformation field with an asymptotic expansion around a 1D curve, the centerline. Here we expand the deformation gradient, \( \mathbf{F} \), around the centerline of the jet as a power series with respect to the offset vector from the centerline, \( \mathbf{h} = h_0 \mathbf{n} + h_1 \mathbf{b} \). We determine the expansion by the following three assumptions:

1. Deformation gradient is approximated to the first order in \( \mathbf{h} \):
2. The cross-section normal and flat to the centerline remains normal and flat; and
3. The deformation in the \( \mathbf{n}, \mathbf{b} \) plane is isotropic.

Assumption 1 implies a first order asymptotic theory. We use \( \mathcal{O}(h) \) to represent any term that is first order in \( \mathbf{h} \), and \( o(h) \) to represent any term that is higher order in \( \mathbf{h} \). The theory is valid when \( \mathbf{h} \) is small (i.e., the jet is thin) by some dimensionless criteria. As we will later identify, there are two criteria for a jet to be considered thin. The first requires the jet to be weakly bended or twisted \( |\mathbf{k} \times \mathbf{h}| \ll 1 \). The second requires the stretch of the jet to be relatively homogeneous, \( \partial \lambda / \partial s |\mathbf{h}| \ll 1 \), where \( \lambda \) is the stretch of the material on the centerline relative to the reference state. The second criterion is required only when the deformation of the beam is dominated by twisting. Assumption 2 follows the classical Euler–Bernoulli beam theory. It approximate the rotation of a material cross-section by the rotation at the centerline. Assumption 3 specifies the in-plane deformation on the cross-section. Under common processing condition, the polymer jet can be treated as incompressible [49]. As we will later show, when the material incompressibility is enforced, the three assumptions completely determine the deformation gradient anywhere in the jet.

It is easier to work with the deformation gradient in the moving local frame than in the fixed lab frame. Introduce the decomposition

\[
\mathbf{F} = \mathbf{R} \mathbf{F}^*. \tag{3.1}
\]

Here \( \mathbf{F} \) is the deformation gradient in the lab frame, \( \mathbf{F}^* \) is the deformation gradient in the local frame. \( \mathbf{R} \) is the rotation of the local frame relative to the lab frame. For the deformation gradient in the local frame, introduce the decomposition

\[
\mathbf{F}^*(s, h_0, h_1, t) = \mathbf{H}(s, h_0, h_1, t) \mathbf{F}^*(s, 0, 0, t). \tag{3.2}
\]

Here \( \mathbf{F}^* \) is the deformation gradient on the centerline. We call \( \mathbf{H} \) the deformation gradient relative to the centerline. For the simplicity of notation, we use the following shorthand notation...
one on the centerline, \( \lambda \) must take the form

\[
F^*(h) := F^*(s, h_o, h_b, t). \tag{3.3}
\]

Following the three assumptions, \( F^*(0) \) must take the form

\[
F^*(0) = \begin{bmatrix}
\frac{1}{\sqrt{\lambda}} \\
1 \\
\frac{1}{\sqrt{\lambda}}
\end{bmatrix}.
\tag{3.3}
\]

Here \( \lambda \) is the stretch of the centerline with respect to the reference state. The jet contracts by \( 1/\sqrt{\lambda} \) in directions perpendicular to the centerline due to incompressibility.

Following the three assumptions \( H \) must take the form

\[
H = \begin{bmatrix}
1 + \kappa \times h \cdot t \\
\kappa \times h \cdot n - \frac{1}{2} \kappa \times h \cdot t \\
\kappa \times h \cdot b - \frac{1}{2} \kappa \times h \cdot b
\end{bmatrix} + o(h). \tag{3.4}
\]

Consider two identical vectors parallel to \( t \) in the reference state, one on the centerline, \( l_o \), and the other offset by \( h_b, h_o \), in (3.4) \( \kappa \times h \cdot t \) describes the relative compression/stretching of \( l_o \) relative to the \( l_o \) due to bending, Fig. 1B. The terms \( \kappa \times h \cdot n \) and \( \kappa \times h \cdot b \) describe the tilting of \( l_o \) relative to \( l_o \) due to twisting, Fig. 1C. The first column can be obtained based on Assumption 2 in the same way as in the elementary beam theory \([45]\). The isotropic in plane expansion/contraction of the cross-section, \( -\frac{1}{2} \kappa \times h \cdot t \), follows from Assumption 3. It enforces material incompressibility to \( O(h) \).

Combining (3.2)-(3.4) we have the expression

\[
F^*(h) = \begin{bmatrix}
\lambda (1 + \kappa \times h \cdot t) \\
\frac{1}{\sqrt{\lambda}} (1 - \frac{1}{2} \kappa \times h \cdot t) \\
\lambda \kappa \times h \cdot b - \frac{1}{2} \lambda \kappa \times h \cdot b
\end{bmatrix} + o(h). \tag{3.5}
\]

Here \( F^*(h) \) is completely expressed in terms of the deformation of the centerline, \( \lambda, \kappa, \) and the offset from the centerline \( h \).

### 3.3. Choice of centerline

Assumptions 2 and 3 are specific to the choice of the centerline. One may then ask how the beam model differs if we choose a different centerline. In the supplementary materials Section 2, we show that the choice of centerline changes the model at \( O(h^2) \) or higher order. Since our beam theory is asymptotic to the \( O(h) \) order, it is independent of the choice of the centerline. For the asymptotic theory to be valid, we require the difference at \( O(h^2) \) level caused by choosing a different centerline is negligible. This requirement gives our two criteria for a jet to be considered thin:

1. \( |\kappa \times h| \ll 1 \); and
2. \( \partial \lambda/\partial s |h| \ll 1 \).

The first criterion implies weak bending or twisting. The second criterion implies nearly homogeneous stretching, where \( \lambda \) is the stretch of the material on the centerline. The second criterion is required only when the deformation of the beam is dominated by twisting.

### 3.4. Velocity gradient

Viscoelasticity is a rate dependent behavior. Rate of deformation is often described by the velocity gradient \([48]\). The velocity gradient is related to the deformation gradient by \([48]\)

\[
L = FF^{-1}. \tag{3.6}
\]

Here a dot on top means the material derivative of the quantity. Using the expression of deformation gradient in the last section, we obtain the velocity gradient in the local frame as (supplementary materials Section 3)

\[
L^*(h) = L^*(0) + \delta L^* + o(h). \tag{3.7}
\]

Here \( L^*(0) \) is the \( O(1) \) term and \( \delta L^* \) is the \( O(h) \) term, with the expressions given in Box 1. Here \( u \) is the velocity of the jet along the centerline. Eqs. (3.7)-(3.9) completely express \( L^*(h) \) in terms of geometric quantities of the centerline, \( \partial u/\partial s, \kappa, \omega \), and the offset from the centerline \( h \).
4. Constitutive relation of a polymer jet

Section 3 expresses the deformation gradient $F$ and the velocity gradient $L$ anywhere in the jet based on the shape and motion of the centerline, $v, \omega$. The stress field in the jet can be calculated using any material model. Since the kinematics has an intrinsic error of $o(h)$, propagating the error through the material model, the stress field is expected to have an error of $o(h)$ as well. Once we have the stress field, the force and torque in the jet can be calculated. Here we demonstrate this procedure with the Oldroyd-B model [46,47].

4.1. Oldroyd-B viscoelastic model

Polymer jets are made of viscoelastic polymer solutions or melts. The deformation of the polymer jet can be decomposed into two parts, the viscous deformation and the elastic deformation. The viscous deformation corresponds to the sliding between polymer chains without changing the chain configuration. The elastic deformation corresponds to the stretching of the polymer chain while keeping the relative position of the polymer chains fixed, Fig. 2A. The deformation gradient can be decomposed into the viscous part and the elastic part correspondingly [50].

$$L^*(0) = \begin{bmatrix} \frac{\partial u}{\partial s} - \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\partial u}{\partial s} \end{bmatrix}$$, and

$$\delta L^* = \begin{bmatrix} \frac{\partial \omega}{\partial s} \times h \cdot n - \frac{1}{2} \left( \frac{D^*F}{Dt} - \frac{1}{2} \frac{\partial u}{\partial s} \right) \times h \cdot t \\ \frac{\partial \omega}{\partial s} \times h \cdot b - \frac{1}{2} \left( \frac{D^*F}{Dt} - \frac{1}{2} \frac{\partial u}{\partial s} \right) \times h \cdot t \end{bmatrix}$$.

Box I.

Here $\eta$ is the viscosity of the solvent.

According to Fig. 2B, we require

$$\sigma_{e} = \sigma_{s}.$$ (4.5)

We assumed that the polymer jet is incompressible. The total stress $\sigma$ has the expression

$$\sigma = \sigma_{e} + \sigma_{s} + pI.$$ (4.6)

Here $p$ is a hydrostatic pressure applied on the material. (4.2)–(4.6) defines the Oldroyd-B model.

Oldroyd-B model is frame indifferent, so (4.2)–(4.6) take the same form in the local frame. Eqs. (4.2)–(4.6) can be simplified to two equations:

$$\mu(B^*_e - I) = -\zeta \left( \frac{D^*B^*_e}{Dt} - L^*B^*_e - B^*_eL^T \right),$$ (4.7)

$$\sigma^* = \mu(B^*_e - I) + \eta(L^* + L^T) + pI.$$ (4.8)

Here $B^*_e = F^*_e F^*_e^T$ is the left Cauchy–Green tensor of the elastic deformation in the local frame. $D^*B^*_e / Dt = \dot{B}_e - WB_e^* + B_e^*W$. The right hand side of (4.7) is the upper-convected derivative of $B_e$ [51], characterizing the changing rate of $B_e$ in a frame following the moving and deforming of material. If we cancel out $B_e$ from (4.7)–(4.8), we will reach the commonly used form of Oldroyd-B model in terms of the upper-convected derivative of stress (supplementary materials Section 4).

4.2. The beam model of an Oldroyd-B jet

We obtain the beam model by combining the kinematics in Section 3.4 and the Oldroyd-B model. Substitute the expression of $L^*(3.7)$ into (4.7), one can verify that $B^*_e$ also has an intrinsic error of $o(h)$. Expand $B^*_e$ and we get

$$B^*_e(h) = B^*_e(0) + \delta B^*_e(h) + o(h).$$ (4.9)

Here $\delta B^*_e(h)$ represents the first order term. At the zeroth order, Oldroyd-B model (4.7) gives

$$\frac{D^*B^*_e(0)}{Dt} - L^*(0)B^*_e(0) - B^*_e(0)L^T(0) = -\frac{\mu}{\zeta} (B^*_e(0) - I).$$ (4.10)

Here the left-hand-side is the upper-convected derivative of $B_e$ on the centerline. At the first order, (4.7) yields

$$\frac{D^*\delta B^*_e(0)}{Dt} - L^*(0)\delta B^*_e - \delta B^*_eL^T(0) = -\frac{\mu}{\zeta} \delta B^*_e + CB^*_e(0) + B^*_e(0)\delta L^T.$$ (4.11)
Here the left-hand-side is not an upper-convected derivative since \( L^* \) is the basis of the derivation, and \( \partial \frac{\partial L^*}{\partial \phi} \) is generally offset from the centerline.

In general, matrix Eqs. (4.10)-(4.11) are twelve independent equations for the six components of \( B^* \) and the six components of \( \delta B^* \). In practice, material only has finite memory of its loading history. When all the memory of the reference state is forgotten, \( B^* \) and \( \delta B^* \) take the following forms:

\[
B^*_e (0) = \begin{bmatrix}
\lambda^2_{\epsilon_e} & \lambda^2_{\epsilon_e} & \lambda^2_{\epsilon_e} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

(4.12)

\[
\delta B^*_e (h) = \begin{bmatrix}
\beta \cdot h \cdot t & \beta \cdot h \cdot n & \beta \cdot h \cdot b \\
\beta \cdot h \cdot n & \beta \cdot h \cdot b & \beta \cdot h \cdot t \\
\beta \cdot h \cdot b & \beta \cdot h \cdot t & \beta \cdot h \cdot n
\end{bmatrix}.
\]

(4.13)

Here \( \lambda^2_{\epsilon_e}, \lambda^2_{\epsilon_e}, \beta, \chi \) are unknowns, \( \lambda^2_{\epsilon_e} \) and \( \lambda^2_{\epsilon_e} \) are the stretch of the polymer chain in the direction parallel to the centerline and perpendicular to the centerline. Note that the stretch of the polymer chain is generally different from the stretch of the jet due to the presence of viscous relaxation. \( \beta, \chi \) characterize the gradient of the stretch of the polymer chains across the cross-section. In this case, (4.10) consists of two equations for \( \lambda^2_{\epsilon_e}, \lambda^2_{\epsilon_e}, \beta, \chi \) at \( (4.10)-(4.11) \) can be solved given the material parameters \( \mu, \kappa \) and the geometric quantities of the centerline, \( \partial u/\partial s, k, \omega \). We assume (4.12) and (4.13) to be true for the rest of the discussion.

As \( B^*_e (h) \) is determined through (4.9), (4.12)-(4.13) to 0 \( (h) \), stress can be determined by the Oldroyd-B model (4.8) to 0 \( (h) \), with an unknown hydrostatic pressure, \( p \). To determine the hydrostatic pressure, we assume that the surface of the jet is stress-free, which implies \( \sigma_{mn} = \sigma_{nb} = 0 \). Consequently

\[
p = \eta \frac{\partial u}{\partial s} - \mu (\lambda^2_{\epsilon_e} - 1) + \left( \eta \left( \frac{D^4 \kappa}{D t^4} - \frac{1}{2} \frac{\partial u}{\partial s} \right) \right) \quad (4.14)
\]

\[
\times h \cdot t + o (h).
\]

The traction on a cross-section normal to the centerline can now be calculated. We get

\[
\sigma (h) = \left[ \beta \left( \mu \lambda^2_{\epsilon_e} - \lambda^2_{\epsilon_e} \right) + 3 \eta \frac{\partial u}{\partial s} \right] + \left[ \begin{bmatrix} m_{\text{bend}} \times h \cdot t \\ m_{\text{twist}} \times h \cdot n \\ m_{\text{twist}} \times h \cdot b \end{bmatrix} \right] + o (h),
\]

(4.15)

Here \( m_{\text{bend}} = \mu (1 - \chi) \beta + 3 \eta \left( \frac{D^4 \kappa}{D t^4} - \frac{1}{2} \frac{\partial u}{\partial s} \right) \) and \( m_{\text{twist}} = \mu \beta + \eta \frac{\partial u}{\partial s} \). The traction (4.15) can be integrated to obtain the total force and torque in the jet. Here it is convenient to locate the centerline at the centroid of the cross-section so that the integration of odd order terms of \( h \) cancels out. The total force and torque on the cross-section can be integrated by \( f = \int_A \sigma (h) da \) and \( M = \int_A (\sigma (h) \times h) da \). Here \( A \) is the cross-section area normal to the centerline. As discussed in Section 3.3, the choice of the centerline is arbitrary, we locate the centerline at the centroid of the cross-section so that the integration of odd order terms of \( h \) cancels out.

\[
f = \left( \mu A \left( \frac{D^4 \kappa}{D t^4} - \frac{1}{2} \frac{\partial u}{\partial s} \right) \right) \quad (4.16)
\]

\[
M = \left[ \begin{bmatrix} m_{\text{bend}} \times t \\ m_{\text{twist}} \cdot t \end{bmatrix} \right] \quad (4.17)
\]

Here \( M = \int_A (h \cdot h) da, m_{\text{bend}} = \int_A (h \cdot b)^2 da, m_{\text{twist}} = \int_A (h \cdot n)^2 da, \) and \( m_{\text{bend}} = \int_A (h \cdot b) da \) constitutes the tensor of the second moment of area of the cross-section.

5. Conservation laws

Take a control volume bounded by the surface of the jet and two cross-sections normal to the centerline located at \( s \) and \( s' \). The conservation of mass requires

\[
\frac{\partial}{\partial t} \int V \rho dV + \int_{A(s')} (v \cdot t) \rho dA - \int_{A(s)} (v \cdot t) \rho dA = \int_{\Gamma} m d s.
\]

(5.1)

Here \( \rho \) is the density of the material, \( m \) is the mass exchange per unit length of the jet, e.g. through the evaporation of the solvent.
Integrate over the cross-section using the velocity distribution 
(S3.10), we get the differential equation
\[
\frac{\partial}{\partial t} \left( \rho \mathbf{v} \right) + \frac{\partial}{\partial s} \left( \rho \mathbf{v} \mathbf{A} \right) = \dot{m}.
\]

(5.2)

The conservation of momentum requires
\[
\frac{\partial}{\partial t} \int_V \rho \mathbf{v} dV + \int_{\partial V} \mathbf{v} \cdot \mathbf{n} dS - \int_{\partial V} \mathbf{v} \cdot \mathbf{f} dS = 0.
\]

(5.3)

Here \( \mathbf{q} \) is the body forces, such as gravity, centrifugal force and Coriolis force. Locate the centerline located at the centroid, the integration of any first order terms over the cross-section vanishes, we have
\[
\frac{\partial}{\partial t} \left( \rho \mathbf{v} \mathbf{A} \right) + \frac{\partial}{\partial s} \left( \rho u \mathbf{v} \mathbf{A} \right) = \mathbf{f} + \mathbf{q}.
\]

(5.4)

The conservation of angular momentum requires
\[
\frac{\partial}{\partial t} \int_V \mathbf{r} \times \rho \mathbf{v} dV + \int_{\partial V} \mathbf{r} \times \mathbf{v} \cdot \mathbf{n} dS - \int_{\partial V} \mathbf{r} \times \mathbf{f} dS = 0.
\]

(5.5)

Locate the centerline at the centroid, the integration of any first order terms over the cross-section vanishes, we get
\[
\frac{\partial}{\partial t} \left( \rho \mathbf{r} \mathbf{J} \right) + \frac{\partial}{\partial s} \left( \rho \mathbf{r} u \mathbf{J} \right) = \mathbf{t} \times \mathbf{f} + \frac{\partial}{\partial s} \left( \rho \mathbf{r} \mathbf{M} \right).
\]

(5.6)

Use (5.4) to cancel out the dependence on the absolute position \( \mathbf{r} \), we get
\[
\frac{\partial}{\partial t} \left( \rho \mathbf{J} \right) + \frac{\partial}{\partial s} \left( 2 \rho \mathbf{J} - \mathbf{t} \cdot \mathbf{f} \right) = \mathbf{t} \times \mathbf{f} + \frac{\partial}{\partial s} \left( \rho \mathbf{M} \right).
\]

(5.7)

Note that \( \mathbf{J} \sim O \left( h^4 \right) \), so that \( \mathbf{t} \times \mathbf{f} = O \left( h^6 \right) \). This is a higher order term that is not predicted by the material model (4.16).

6. Rotary jet spinning

Rotary jet spinning is a platform that uses centrifugal force to produce nanofiber. It is praised for its orders-of-magnitude improvement in production rate comparing to conventional electrospinning [11]. Rotary jet spinning consists of a rapidly rotating reservoir where polymer solution/polymer melt is fed in [9,10]. Fig. 3. A few orifices are opened on the side of the reservoir. Under the centrifugal force, the polymer solution is pulled out from an orifice and form a polymer jet. The polymer jet is elongated and bended under centrifugal force, Coriolis force, air drag, and gravity. The elongated polymer jet then solidifies by the evaporation of the solvent [9], cooling below the melting temperature [14], or entering a precipitation bath [13]. Previous studies show that the jet may be acutely bended near the orifice due to a combination of centrifugal force, Coriolis force, and viscous stress, where a string model fails to model rotary jet spinning [31]. While the existing beam model of rotary jet spinning resolves the bending near the orifice, it only studies viscous jet with no viscoelastic relaxation [32]. In this section, we use the beam model developed in previous sections to model the fiber formation in rotary jet spinning.

For simplicity, we only model the steady state of jet in rotary jet spinning. Gravity, air-drag, solvent evaporation, and solidification of the jet are neglected. Under these assumptions, a boundary value problem is formulated using the result derived in the previous sections (supplementary materials 5). The problem is governed by five dimensionless groups: the Reynolds number \( Re = \rho v_0\Omega_0 / \zeta \) characterizing the competition between inertia and the viscoelasticity in the jet, the Rossby number \( Ro = v_0 / \Omega_0 r_0 \) characterizing the effect of centrifugal force and Coriolis force, the Weissenberg number for the polymer part and the solvent part \( Wi_p = \xi \eta_0 / \mu_0 r_0 \) and \( Wi_s = \eta_0 / \mu_0 r_0 \), characterizing the viscoelastic relaxation in the jet, and the slenderness ratio \( Sl = a_0 / r_0 \), comparing the resistance to the bending of the jet versus the resistant to the stretching of the jet.

In Fig. 4, we choose our simulation condition to represent common spinning condition by fixing \( Re = 1, Ro = 0.1, Wi_p = 10^{-2}, Sl = 10^{-3} \) [4], and study the effect of the viscoelasticity of the polymer jet by varying \( Wi_p = 0.01, 0.1, 1 \). When \( Wi_p = 0.01 \), the viscoelastic relaxation is fast comparing to the time scale of deforming. The jet behaves like a viscous fluid. When \( Wi_p = 1 \), the viscoelastic relaxation and the time scale of deforming is comparable. The jet behaves like an elastic solid near the reservoir.

Fig. 4A plots the trajectories of the jets of different \( Wi_p \). As the jet become more elastic, the jet goes closer around the reservoir. In fact, a perfectly elastic jet would fall onto a fast rotating reservoir with \( Ro < 1 \) (supplementary material Section 6). Fig. 4B compares the fiber radius. The highly viscous jet (\( Wi_p = 0.01 \)) experiences constant reduction in radius during the spinning, as the centrifugal force keeps driving the viscous thinning of the jet. On the other hand, the highly elastic jet (\( Wi_p = 1 \)) resists reduction in radius in most range of the jet (\( 10^{-1} < s / r_0 < 10^{1} \)), after an abrupt strongly stretched near the orifice (\( s / r_0 < 10^{-1} \)). This is because that the elasticity can withstand a constant stress without thinning out. The localized stretch is where the elastic deformation happens. In Fig. 4C, we plot the stretch of polymer chains: \( \lambda_{chain} = \sqrt{(\lambda_{||}^2 + 2\lambda_{\perp}^2)} / 3 \) [52]. The stretch of polymer chains is an indicator.
of the microscopic chain alignment, which is desirable in creating ultra-strong fibers [53] or inducing certain protein folding [54]. Fig. 4C shows that viscous jet ($W_I = 0.01$) does not induce any chain stretch as the viscoelastic relaxation is too fast compared to the deformation rate. In contrast, the elastic jet ($W_I = 1$) accumulates chain stretch. In the intermediate case ($W_I = 0.1$), polymer chains are stretched initially ($s/r_0 < 10^0$) when the deformation is rapid. The chain stretch is gradually lost as the jet flies further away from the reservoir ($s/r_0 > 10^0$) when the deformation rate drops.

To probe how the beam bending contributes to the deformation of the jet, we plot the curvature normalized by the jet radius along the jet in Fig. 4D. It shows that strong bending deformation localizes at a small region near the orifice, ($s/r_0 < 0.1$). While the jet also has a finite curvature elsewhere as shown in Fig. 4A, the bending deformation is negligible due to the great reduction in the jet radius (see Fig. 4B). To make sure that the model is valid, we need to satisfy the criteria $|\kappa \times \mathbf{h}| \ll 1$ and $\partial \lambda/\partial s |_{\mathbf{h}} \ll 1$. Since our jet is free of twisting, the second criterion is dropped. Fig. 4D shows that $|\kappa \times \mathbf{h}| \ll 1$ is indeed satisfied.

In summary, these simulations show that viscoelasticity strongly influence the rotary jet spinning process. The more elastic jet is better at align polymer chains but is poorer in reducing the fiber diameter, while the more viscous jet is the converse. It is important to design the spinning condition to achieve intermediate viscoelastic relaxation so that small fiber diameter and polymer chain alignment are achieved at the same time.

7. Conclusion remarks

This paper formulates a first-order beam theory for nonlinear viscoelastic material. The theory generalizes the classical Euler–Bernoulli theory to account for finite deformation and material incompressibility. The kinematics derived is then combined with the Oldroyd-B model to derive the constitutive equations of a nonlinear viscoelastic beam. The beam model is then used to study the viscoelastic relaxation in rotary jet spinning. Our model successfully captures the strong bending near the orifice that fails string models and the highly elastic behavior of the jet that cannot be modeled by existing beam model. Our theory has potential applications in many other manufacturing processes involving polymer jets, such as 3D printing, electro-spinning, and blow spinning.

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Appendix A. Supplementary data

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References
